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FM03

(9665/FM03) Unit FP2 Pure Mathematics

Mark scheme

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Key to mark scheme abbreviations

M	Mark is for method
m	Mark is dependent on one or more M marks and is for method
A	Mark is dependent on M or m marks and is for accuracy
B	Mark is independent of M or m marks and is for method and accuracy
E	Mark is for explanation
√ **or ft**	Follow through from previous incorrect result
CAO	Correct answer only
CSO	Correct solution only
AWFW	Anything which falls within
AWRT	Anything which rounds to
ACF	Any correct form
AG	Answer given
SC	Special case
oe	Or equivalent
A2, 1	2 or 1 (or 0) accuracy marks
–*x* EE	Deduct *x* marks for each error
NMS	No method shown
PI	Possibly implied
SCA	Substantially correct approach
sf	Significant figure(s)
dp	Decimal place(s)
ISW	Ignore subsequent working

Q	Answer	Marks	Comments
1(a)(i)	Rotation [Through an angle of] 90° about the y-axis	**M1** **A1**	Rotation **oe** **SC1** for 'rotate' or 'rotated' or 'rotates' and 90° about the y-axis
		2	

Q	Answer	Marks	Comments
1(a)(ii)	Reflection [In plane] $x = 0$	**M1** **A1**	Reflection $x = 0$ **oe** eg y-z plane **SC1** for 'reflect' or 'reflected' or 'reflects' in $x = 0$
		2	

Q	Answer	Marks	Comments
1(b)	$\mathbf{M}_R\mathbf{M}_S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\mathbf{M}_T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	**M1** **A1**	Correct order of multiplication $\mathbf{M}_T = \mathbf{M}_R\mathbf{M}_S$ and no more than one sign/numerical error in finding $\mathbf{M}_T$ **PI**
		2	

	Question 1 Total	**6**	

Q	Answer	Marks	Comments
2(a)	$\mathbf{a}\times\mathbf{b} = \mathbf{i}+4\mathbf{j}-3\mathbf{k}$	B1	**oe** Column vector
		1	

Q	Answer	Marks	Comments
2(b)	$(\mathbf{a}\times\mathbf{b})\bullet\mathbf{c}=(\mathbf{i}+4\mathbf{j}-3\mathbf{k})\bullet(-3\mathbf{i}+t\mathbf{j}+9\mathbf{k})=0$	M1	Equating a relevant scalar triple product or equivalent determinant to 0 **PI**
	$-3+4t-27=0$	A1ft	Correct evaluation of scalar triple product, using their cross product **ft**
	$t=7.5$	A1	Correct value of t **oe**
		3	

Q	Answer	Marks	Comments
2(c)	$\mathbf{c}=\lambda(\mathbf{a}\times\mathbf{b})\Rightarrow(-3\mathbf{i}+t\mathbf{j}+9\mathbf{k})=\lambda(\mathbf{i}+4\mathbf{j}-3\mathbf{k})$	M1	$\mathbf{c}=\lambda(\mathbf{a}\times\mathbf{b})$ or $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{0}$ used eg $(36+3t)\mathbf{i}+(9-9)\mathbf{j}+(t+12)\mathbf{k}=\mathbf{0}$ or $\mathbf{a}\bullet\mathbf{c}=0$ or $\mathbf{b}\bullet\mathbf{c}=0$
	$[\lambda=-3]\Rightarrow t=-12$	A1	
		2	

	Question 2 Total	6	

Q	Answer	Marks	Comments
3(a)	$\left[x^4 - px^2 + qx - r \equiv (x-\alpha)^3(x-\beta)\right]$		
	$\alpha+\alpha+\alpha+\beta = 3\alpha+\beta = 0$ $\alpha^3\beta = -r$ $\alpha^2+\alpha^2+\alpha^2+\alpha\beta+\alpha\beta+\alpha\beta = 3\alpha^2+3\alpha\beta = -p$	M1 A1	**M1**: At least one of the three correct **A1**: All three correct
	$\beta = -3\alpha$		
	$\Rightarrow p = 6\alpha^2 \quad ; \quad r = 3\alpha^4$	A1	Either correct
	$p^2 = 36\alpha^4 = 12\left(3\alpha^4\right)$		
	$\Rightarrow p^2 = 12r$	A1	**AG** Must be convincingly shown
		4	

Q	Answer	Marks	Comments
3(b)	$\alpha^3+\alpha^2\beta+\alpha^2\beta+\alpha^2\beta = -q$ $-q = \alpha^3+3\alpha^2\beta = \alpha^3-9\alpha^3 \Rightarrow q = 8\alpha^3$	M1	Condone $\alpha^3+\alpha^2\beta+\alpha^2\beta+\alpha^2\beta = +q$
	$pr = 18\alpha^6 = 18\left(\frac{q}{8}\right)^2 = \frac{9}{32}q^2$	A1	**CSO**
		2	

	Question 3 Total	**6**	

Q	Answer	Marks	Comments
4	$y^2 = 4x$; $2y\dfrac{dy}{dx} = 4$ or $\dfrac{dy}{dx} = \dfrac{2}{y}\left(= \dfrac{2}{\sqrt{4x}}\right)$	B1	Correct differentiation of either $y^2 = 4x$ or $y = 2\sqrt{x}$
	[Area= $2\pi\int y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\,dx$]		
	$= 2\pi\int_{[3]}^{[4]} 2x^{0.5}\sqrt{1+\dfrac{1}{x}}\,dx$	M1	Substitution of $y = 2\sqrt{x}$ and their $\dfrac{dy}{dx}$ into $[2\pi]\int y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}\,dx$
	$= 4\pi\int_{[3]}^{[4]}(x+1)^{0.5}\,dx$	A1	
	$= 4\pi\left[\dfrac{2}{3}(x+1)^{1.5}\right]_{[3]}^{[4]}$	M1	Correct integration of $(x+1)^{0.5}$
	$= \dfrac{8\pi}{3}\left(5\sqrt{5} - 4\sqrt{4}\right)$		Must see substitution line before final answer
	$= \dfrac{8\pi}{3}\left(5\sqrt{5} - 8\right)$	A1	**AG** Must be convincingly shown

	Question 4 Total	**5**	

Q	Answer	Marks	Comments
5	$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{1}{x}y = \frac{1}{x}\sin^{-1}x$		
	I.F. is $\left[\mathrm{e}^{\int \frac{1}{x}\,\mathrm{d}x} = \mathrm{e}^{\ln x}\right] = x$	M1	
	$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \sin^{-1}x \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(xy) = \sin^{-1}x$		
	$xy = \int \sin^{-1}x\,\mathrm{d}x$	M1	Multiplying both sides of the differential equation by their I.F. and integrating LHS to get $y \times$ I.F
	$xy = x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$	M1	Attempt at Integration by parts
	$xy = x\sin^{-1}x + \sqrt{1-x^2}\quad [+c]$	A1	**oe** Condone absence of $+c$
	$y = \sin^{-1}x + \frac{\sqrt{1-x^2}}{x} + \frac{c}{x}$	A1	**ACF**

	Question 5 Total	**5**	

Q	Answer	Marks	Comments
6(a)(i)	$z^n = \cos n\theta + \mathrm{i}\sin n\theta$	**M1**	
	$z^{-n} = \cos(-n\theta) + \mathrm{i}\sin(-n\theta)$ $= \cos(n\theta) - \mathrm{i}\sin(n\theta)$	**E1**	As shown in Answer column or using $\frac{1}{\cos n\theta + \mathrm{i}\sin n\theta} \times \frac{\cos n\theta - \mathrm{i}\sin n\theta}{\cos n\theta - \mathrm{i}\sin n\theta}$
	$z^n + \frac{1}{z^n} = 2\cos n\theta$	**A1**	**AG** Must be convincingly shown **M1 E0 A1** is possible eg for those who just quote $z^{-n} = \cos(n\theta) - \mathrm{i}\sin(n\theta)$
		3	

Q	Answer	Marks	Comments
6(a)(ii)	$z - \frac{1}{z} = 2\mathrm{i}\sin\theta$	**B1**	
		1	

Q	Answer	Marks	Comments
6(b)	$\cos^3\theta\sin^4\theta = 2^{-3}\left(z+z^{-1}\right)^3(2\mathrm{i})^{-4}\left(z-z^{-1}\right)^4$	**B1**	
	$2^7\cos^3\theta\sin^4\theta = \left(z+z^{-1}\right)^3\left(z-z^{-1}\right)^4$		
	$=\left(z-z^{-1}\right)\left(z^2-z^{-2}\right)^3$	**M1**	**PI**
	$=\left(z-z^{-1}\right)\left(z^6-3z^2+3z^{-2}-z^{-6}\right)$ $= z^7 - z^5 - 3z^3 + 3z + 3z^{-1} - 3z^{-3} - z^{-5} + z^{-7}$	**A1**	Correct expansion of $\left(z+z^{-1}\right)^3\left(z-z^{-1}\right)^4$
	$=\left(z^7+z^{-7}\right)-\left(z^5+z^{-5}\right)-3\left(z^3+z^{-3}\right)+3\left(z+z^{-1}\right)$	**M1**	Groups terms so as to use **(a) PI**
	$= 2\cos 7\theta - 2\cos 5\theta - 6\cos 3\theta + 6\cos\theta$		
	$\cos^3\theta\sin^4\theta$ $= \frac{3}{64}\cos\theta - \frac{3}{64}\cos 3\theta - \frac{1}{64}\cos 5\theta + \frac{1}{64}\cos 7\theta$	**A1**	
		5	

	Question 6 Total	**9**	

Q	Answer	Marks	Comments
7(a)	$\left[\frac{dy}{dx}=\right]2(1+x)\ln(1+x)+1+x$	M1	Product rule used
	$\left[\frac{d^2y}{dx^2}=\right]2\ln(1+x)+2+1$		Must see before final line
	$\frac{d^2y}{dx^2}=2\ln(1+x)+3$	A1	**AG** Must be convincingly shown
		2	

Q	Answer	Marks	Comments
7(b)	From **(a)**: $\frac{d^3y}{dx^3}\left[=\frac{d}{dx}\left(2\ln(1+x)+3\right)\right]=\frac{2}{1+x}$ When $n=3$, the given formula becomes $\frac{d^3y}{dx^3}=\frac{(-1)^{3-1}2(3-3)!}{(1+x)^{3-2}}=\frac{(-1)^2 2(0)!}{(1+x)^1}=\frac{2}{(1+x)}$ [So formula is true for $n=3$]	B1	Correct work to show formula true for $n=3$
	Assume formula true for $n=k$ **(*)**, ie $\frac{d^ky}{dx^k}=\frac{(-1)^{k-1}2(k-3)!}{(1+x)^{k-2}}$ so $\frac{d^{k+1}y}{dx^{k+1}}\left[=\frac{d}{dx}\left(\frac{d^ky}{dx^k}\right)\right]=\frac{d}{dx}\left(\frac{(-1)^{k-1}2(k-3)!}{(1+x)^{k-2}}\right)$	M1	Assumes formula true for $n=k$ and considers $\frac{d}{dx}\left(\frac{(-1)^{k-1}2(k-3)!}{(1+x)^{k-2}}\right)$
	$=\frac{d}{dx}\left((-1)^{k-1}2(k-3)!(1+x)^{-(k-2)}\right)$ $=(-1)^{k-1}2(k-3)!\frac{d}{dx}\left((1+x)^{-(k-2)}\right)$ $=(-1)^{k-1}2(k-3)!(-1)(k-2)(1+x)^{-(k-2)-1}$	A1	

	$=(-1)^{k+1-1}\,2(k-2)!(1+x)^{-(k+1-2)}$ $=\dfrac{(-1)^{k+1-1}\,2(k+1-3)!}{(1+x)^{k+1-2}}$	A1	Must be convincingly shown
	Hence formula is true for $n = k+1$ **(**)** and since true for $n = 3$ **(***)**, formula $\dfrac{d^n y}{dx^n} = \dfrac{(-1)^{n-1}\,2(n-3)!}{(1+x)^{n-2}}$ is true for integers $n \geq 3$ by **induction (****)**	E1	Must have **(*)** , **(**)**, **(***)**, present, previous 4 marks scored and a final statement **(****)** clearly indicating that it relates to positive integers ≥ 3 by induction Condone n = 3, 4, 5….
		5	

Q	Answer	Marks	Comments
7(c)	$\dfrac{(-1)^9\,2(10-3)!}{(1+0)^8\,10!}+\dfrac{(-1)^{10}\,2(11-3)!}{(1+0)^9\,11!}$	M1	**PI** correct answer
	$\left[-\dfrac{1}{360}+\dfrac{1}{495}\right]=-\dfrac{1}{1320}$	A1	
		2	

	Question 7 Total	**9**	

Q	Answer	Marks	Comments
8	Aux. equation $m^2+2m=0$; $m=0,-2$	**M1**	Forming and solving the correct aux. equation, getting at least $m=0$ **PI**
	$y_{CF}=A+Be^{-2x}$	**A1**	Correct CF
	$y_{PI}=ax^2+bx+C\cos x+D\sin x$	**B1; B1**	$y_{PI}=ax^2+bx$ (**B1**) $+C\cos x+D\sin x$ (**B1**) seen or used
	$y'_{PI}=2ax+b-C\sin x+D\cos x$ $y''_{PI}=2a-C\cos x-D\sin x$ $2a-C\cos x-D\sin x+4ax+2b+2D\cos x$ $-2C\sin x=12x+4+10\sin x$	**M1**	Their y'_{PI} and y''_{PI} both substituted into the differential equation
	$4a=12\Rightarrow a=3$; $2a+2b=4\Rightarrow b=-1$; $-2C-D=10$; $-C+2D=0\Rightarrow C=-4, D=-2$	**A1**	At least two correct
	$y_{PI}=3x^2-x-2\sin x-4\cos x$	**A1**	Correct particular integral
	$[y_{GS}=]\ A+Be^{-2x}+3x^2-x-2\sin x-4\cos x$	**B1ft**	Their CF + their PI with exactly two arbitrary constants
	When $x=0$, $\frac{dy}{dx}=5\Rightarrow -2B-1-2=5,\ \Rightarrow B=-4$ When $x=0$, $y=1\Rightarrow A+B-4=1$, $A=9$	**A1**	Either $A=9$ or $B=-4$
	$y=9-4e^{-2x}+3x^2-x-2\sin x-4\cos x$	**A1**	

	Question 8 Total	**10**	

Q	Answer	Marks	Comments
9	$\int \frac{1+2x}{x+x^2}\,dx = \ln\left(x+x^2\right)\ [+c]$	B1	
	$\int \frac{\sin 2x}{1-\cos 2x}\,dx = \frac{1}{2}\ln\left(1-\cos 2x\right)\ [+c]$	B1	
	$\int_0^{\frac{\pi}{4}}\left(\frac{1+2x}{x+x^2}-\frac{\sin 2x}{1-\cos 2x}\right)dx$ $=\lim_{a\to 0}\int_a^{\frac{\pi}{4}}\left(\frac{1+2x}{x+x^2}-\frac{\sin 2x}{1-\cos 2x}\right)dx$	E1	Evidence of limit 0 having been replaced by a (oe) at any stage and $\lim_{a\to 0}$ seen or taken
	$=\lim_{a\to 0}\left[\ln\left(x+x^2\right)-\frac{1}{2}\ln\left(1-\cos 2x\right)\right]_a^{\frac{\pi}{4}}=$ $\ln\left(\frac{\pi}{4}+\frac{\pi^2}{16}\right)-\lim_{a\to 0}\left(\ln\left(a+a^2\right)-\frac{1}{2}\ln\left(1-\cos 2a\right)\right)$	M1	Substitution of limits in a correct manner
	$=\ln\left(\frac{\pi}{4}+\frac{\pi^2}{16}\right)-\lim_{a\to 0}\left(\frac{1}{2}\ln\left(\frac{\left(a+a^2\right)^2}{1-\cos 2a}\right)\right)=$	M1	Expressing terms in a **(oe)** as a single ln term at some stage
	$\ln\left(\frac{\pi}{4}+\frac{\pi^2}{16}\right)-\lim_{a\to 0}\left(\frac{1}{2}\ln\left(\frac{\left(a^2+2a^3+a^4\right)}{1-\left(1-2a^2+\frac{16a^4}{24}+\ldots\right)}\right)\right)$	B1	Correct series for $\cos 2a$ as far as a^4 Accept $\cos 2a = 1-2a^2+0\left(a^4\right)$ or Correct series for $\sin a$ as far as a^3 Accept $\sin a = a+0\left(a^3\right)$
	$=\ln\left(\frac{\pi}{4}+\frac{\pi^2}{16}\right)-\lim_{a\to 0}\left(\frac{1}{2}\ln\left(\frac{1+2a+a^2}{2-\frac{2}{3}a^2+\ldots}\right)\right)$	m1	Writing the expression involving a terms in a form so that the $\lim_{a\to 0}\left(f(a)\right)$ is defined
	$=\ln\left(\frac{\pi}{4}+\frac{\pi^2}{16}\right)-\frac{1}{2}\ln\left(\frac{1}{2}\right)=\ln\left(\frac{\sqrt{2}\,\pi\left(4+\pi\right)}{16}\right)$	A1	**ACF**

	Question 9 Total	**8**	

Q	Answer	Marks	Comments
10(a)	$\mathbf{d} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$; $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$	**B1**	Correct **d** and correct **n**, seen or used, where **d** is direction vector of L and **n** is direction vector of the normal to Π_1
	$\mathbf{d} \bullet \mathbf{n} = 2 - 1 + 12 \; [= 13]$	**M1**	Finds a numerical expression for the scalar product of their **d** and their **n**
	$\sqrt{2^2+1^2+4^2}\,\sqrt{1^2+1^2+3^2}\cos\theta = 13$ $\cos\theta = \dfrac{13}{\sqrt{231}}$	**m1**	**ft** their **d** and their **n**
	Acute angle between L and Π_1 $= 90° - \cos^{-1}\left(\dfrac{13}{\sqrt{231}}\right) = 58.8°$	**A1**	**CAO**
		4	

Q	Answer	Marks	Comments
10(b)	$\overrightarrow{OA} = \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix}$	**B1**	Finds a point on L **PI**
	$\overrightarrow{AP} = \pm\begin{bmatrix} -3 \\ 0 \\ 12 \end{bmatrix}$ or $\overrightarrow{AP} = \pm\begin{bmatrix} -3-2\lambda \\ \lambda \\ 12-4\lambda \end{bmatrix}$	**M1**	Finds the vector between their point and P or Finds the vector between their general point on L and P
	$\overrightarrow{AP} \times \mathbf{d} = \begin{bmatrix} -3 \\ 0 \\ 12 \end{bmatrix} \times \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 36 \\ 3 \end{bmatrix}$ or $0 = \overrightarrow{AP} \bullet \mathbf{d} = \begin{bmatrix} -3-2\lambda \\ \lambda \\ 12-4\lambda \end{bmatrix} \bullet \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = -42 + 21\lambda$ $\lambda = 2$	**M1**	Obtains the vector product of their $\overrightarrow{AP}$ and their direction vector or Sets the scalar product of their $\overrightarrow{AP}$ and their direction vector = 0 to form a linear equation

	$\left\|\overrightarrow{AP}\times\mathbf{d}\right\| = 3\sqrt{161}$ $\|\mathbf{d}\| = \sqrt{21}$ Shortest distance $= \dfrac{\left\|\overrightarrow{AP}\times\mathbf{d}\right\|}{\|\mathbf{d}\|} = \dfrac{3\sqrt{161}}{\sqrt{21}} = \sqrt{69}$ or Shortest vector $= \begin{bmatrix}-7\\2\\4\end{bmatrix}$ Shortest distance $= \sqrt{69}$	**A1**	**CAO**
		4	

Q	Answer	Marks	Comments
10(c)	$\mathbf{b} = \begin{bmatrix}1\\1\\3\end{bmatrix} \times \begin{bmatrix}1\\-3\\2\end{bmatrix} = \begin{bmatrix}11\\1\\-4\end{bmatrix}$	**M1** **A1**	**oe** At least two components correct **oe** A correct **b**
	$\Pi_1 : x+y+3z=2$; $\Pi_2 : x-3y+2z=1$ eg $(s, 0, t)$ solving simultaneously $s+3t=2$ and $s+2t=1$	**M1**	Valid method to find a common point
	Common point $(-1, 0, 1)$	**A1**	Any correct common point eg $\left(0, \frac{1}{11}, \frac{7}{11}\right)$, $\left(\frac{7}{4}, \frac{1}{4}, 0\right)$
	Equation of line of intersection of Π_1 and Π_2 $\left(\mathbf{r} - \begin{bmatrix}-1\\0\\1\end{bmatrix}\right) \times \begin{bmatrix}11\\1\\-4\end{bmatrix} = \mathbf{0}$	**A1**	**oe** but must be in the form $(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = \mathbf{0}$ eg $\left(\mathbf{r} - \left(\frac{7}{4}\mathbf{i} + \frac{1}{4}\mathbf{j}\right)\right) \times (11\mathbf{i} + \mathbf{j} - 4\mathbf{k}) = \mathbf{0}$
		5	

	Question 10 Total	**13**	

Q	Answer	Marks	Comments
11(a)	Cofactor matrix $\begin{bmatrix} c-11 & -23 & -1-2c \\ -2 & -3 & 1 \\ -2c & 13 & c \end{bmatrix}$	M1 A2	One complete row or column or diagonal correct All nine entries correct else **A1** for at least six entries correct
	det $\mathbf{M} = -3c - 13$ Inverse matrix $\mathbf{M}^{-1} =$	M1	Transpose of their cofactors with no more than one further error **and** division by their det $\mathbf{M}\left(\neq 0\right)$
	$= \dfrac{1}{-3c-13}\begin{bmatrix} c-11 & -2 & -2c \\ -23 & -3 & 13 \\ -1-2c & 1 & c \end{bmatrix}$	A1	A correct $\mathbf{M}^{-1}$ scores 5 marks
		5	

Q	Answer	Marks	Comments
11(b)(i)	$\det(\mathbf{M} - \lambda \mathbf{I}) = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 1 & c-\lambda & -11 \\ 2 & -1 & 1-\lambda \end{vmatrix}$ $(1-\lambda)\left((c-\lambda)(1-\lambda)-11\right)+2\left(-1-2(c-\lambda)\right)$	M1	Sets up and expands $\det(\mathbf{M} - \lambda \mathbf{I})$ to remove all determinants
	$-\lambda^3 + (2+c)\lambda^2 - (2c-14)\lambda - 3c - 13 = 0$	m1 A1	Forms a cubic polynomial in λ Correct cubic equation
	$2c - 14 = -4$; $2c = 10$	M1	Uses $\dfrac{C}{A} = -4$ ft their cubic equation
	$c = 5$	A1	**AG** Must be convincingly shown
		5	

Q	Answer	Marks	Comments
11(b)(ii)	$\lambda^3 - 7\lambda^2 - 4\lambda + 28 = 0$	M1	Substitutes $c = 5$ into their cubic equation and solves to find three real distinct values of λ
	$\lambda = -2,\ 2,\ 7$	A1	Correct three values
		2	

Q	Answer	Marks	Comments
11(b)(iii)	$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 5 & -11 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$; $x + 2z = -2x$ $x + 5y - 11z = -2y$ $2x - y + z = -2z$ $3x + 2z = 0$ $x + 7y - 11z = 0$ $2x - y + 3z = 0$	M1	**M**$\mathbf{v} = \lambda_{\text{least}}$ **v oe** and attempt to form a system of equations, **ft** their least value of λ
	$z = \dfrac{-3x}{2}$ $y = \dfrac{-5x}{2}$	A1ft	Eliminates so that each of two variables are written solely in terms of the third, with at least one correct
	$x = 2,\ y = -5,\ z = -3$; $\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}$	A1	An eigenvector in form $\beta\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \beta \neq 0$
		3	

	Question 11 Total	**15**	

Q	Answer	Marks	Comments
12(a)	$4-4\mathrm{i} = 4\sqrt{2}\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}\right)}$	B1	Correct modulus and argument **PI** a correct root
	$z^5 = 4\sqrt{2}\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}+2n\pi\right)}$		
	$z = \sqrt{2}\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{20}+\frac{2n\pi}{5}\right)}$ [for $n = 0, \pm1, \pm2$]	M1	If not stated in the general form award **M1** for at least one correct root of the equation.
	[Root closest to the Imaginary axis is] $\sqrt{2}\,\mathrm{e}^{\mathrm{i}\left(-\frac{9\pi}{20}\right)}$	A1	**CAO**
		3	

Q	Answer	Marks	Comments
12(b)	[Roots above the Real axis are] $\sqrt{2}\,\mathrm{e}^{\mathrm{i}\left(\frac{7\pi}{20}\right)}$ and $\sqrt{2}\,\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{4}\right)}$	M1	**oe** **ft** their roots of $z^5 = 4-4\mathrm{i}$ but only if exactly two roots are above the Real axis and inequalities are satisfied
	$\left[\text{Product} = 2\,\mathrm{e}^{\mathrm{i}\left(\frac{22\pi}{20}\right)}\right]$ $2\,\mathrm{e}^{\mathrm{i}\left(-\frac{9\pi}{10}\right)}$	A1	**oe**
		2	

Q	Answer	Marks	Comments
12(c)	[Length of each side of the pentagon is] $2\times\sqrt{2}\sin\left(\frac{\pi}{5}\right)$	M1	**PI**
	[Perimeter of the pentagon is] $10\sqrt{2}\sin\left(\frac{1}{5}\pi\right)$	A1	In the form $a\sin(b\,\pi)$ with $a = 10\sqrt{2}$ **oe** exact form and $b = 0.2$ **oe** exact form
		2	

	Question 12 Total	**7**	

Q	Answer	Marks	Comments
13(a)	$x^2 - y^2 = 8$; $r^2\cos^2\theta - r^2\sin^2\theta = 8$	M2,1	**M2**:Correctly uses two of $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$ If not **M2** then award **M1** if only uses one of the three correctly
	$r^2\left(\cos^2\theta - \sin^2\theta\right) = 8$; $r^2\cos 2\theta = 8$ $r^2 = \dfrac{8}{\cos 2\theta} = 8\sec 2\theta$; $r = \sqrt{8\sec 2\theta}$	A1	**AG** Must be convincingly shown
		3	

Q	Answer	Marks	Comments
13(b)	At A, $[r = 4]$; $\tan\theta = \dfrac{y}{x} = \dfrac{2}{2\sqrt{3}} = \dfrac{1}{\sqrt{3}}$		
	$\theta = \dfrac{\pi}{6}$	B1	Seen or used
	[Area of R] = $\dfrac{1}{2}\int_{[0]}^{\left[\frac{\pi}{6}\right]}\left(\sqrt{8\sec 2\theta}\right)^2[\mathrm{d}\theta]$	M1	Use of $\dfrac{1}{2}\int r^2[\mathrm{d}\theta]$
	$= 4\int_{[0]}^{\left[\frac{\pi}{6}\right]}\sec 2\theta\ [\mathrm{d}\theta]$	A1	**oe**
	$\int \sec 2\theta\,[\mathrm{d}\theta] = \dfrac{1}{2}\ln(\sec 2\theta + \tan 2\theta) + [c]$ [Area of R] $= \left[2\ln(\sec 2\theta + \tan 2\theta)\right]_{[0]}^{\left[\frac{\pi}{6}\right]}$	M1	$k\ln(\sec 2\theta + \tan 2\theta)$ **oe**
	$= 2\ln\left(\sec\left(\dfrac{\pi}{3}\right) + \tan\left(\dfrac{\pi}{3}\right)\right)$	A1	**oe**
	$2\ln\left(2+\sqrt{3}\right) = \ln\left(2+\sqrt{3}\right)^2$		
	$\ln\left(7+4\sqrt{3}\right)$	A1	
		6	

Q	Answer	Marks	Comments
13(c)	[Required area] = Area of triangle *OAB* $-2\times$ Area of *R* $=\frac{1}{2}\left(2\sqrt{3}\right)\left(4\right)-2\times\ \ln\left(7+4\sqrt{3}\right)$	**M1**	$=\frac{1}{2}\left(2\sqrt{3}\right)\left(4\right)-2\times$ their Area of *R* **oe** eg $\frac{1}{2}(4)^2\sin\left(\frac{\pi}{3}\right)-2\times\ln\left(7+4\sqrt{3}\right)$
	$=4\sqrt{3}-2\ln\left(7+4\sqrt{3}\right)$	**A1**	**oe** exact answer eg $4\sqrt{3}-4\ln\left(2+\sqrt{3}\right)$ $4\sqrt{3}-\ln\left(97+56\sqrt{3}\right)$
		2	

	Question 13 Total	**11**	

Q	Answer	Marks	Comments
14	$y = 7\cosh x - \sinh x - 4x + 4\tanh^{-1} k$ [Let A to be the point on C where the horizontal tangent $y = 10$ touches the curve C]		
	$\frac{dy}{dx} = 7\sinh x - \cosh x - 4$	**B1**	**oe**
	$\frac{dy}{dx} = 0$ at A $\Rightarrow \frac{7}{2}\left(e^x - e^{-x}\right) - \frac{1}{2}\left(e^x + e^{-x}\right) - 4 = 0$	**M1**	Correctly expresses $\sinh x$ and $\cosh x$ in terms of exponentials and equates their $\frac{dy}{dx}$ to 0
	$3e^{2x} - 4e^x - 4 = 0$	**m1**	Forms a quadratic equation in e^x
	$e^x = 2$, $e^x \neq -\frac{2}{3}$	**A1**	From correct quadratic equation gets $e^x = 2$ and indicates rejection of a negative value.
	$\Rightarrow x = \ln 2$	**A1**	Correct x-coordinate of A
	$10 = 7\cosh(\ln 2) - \sinh(\ln 2) - 4(\ln 2) + 4\tanh^{-1} k$	**M1**	**ft** their x-coordinate of A **PI**
	$4\tanh^{-1} k = 4\ln 2 + 2$	**A1**	**oe**
	$4\left(\frac{1}{2}\ln\left(\frac{1+k}{1-k}\right)\right) = 4\ln 2 + 2$	**M1**	Uses $\tanh^{-1} k = \frac{1}{2}\ln\left(\frac{1+k}{1-k}\right)$ **oe**
	$\ln\left(\frac{1+k}{1-k}\right) = 2\ln 2 + \ln e = \ln(4e)$	**m1**	
	$\frac{1+k}{1-k} = 4e$		
	$k = \frac{e - 0.25}{e + 0.25}$	**A1**	**ACF**

	Question 14 Total	**10**	