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FURTHER MATHEMATICS

FM03

(9665/FM03) Unit FP2 Pure Mathematics

Mark scheme

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Key to mark scheme abbreviations

M	Mark is for method
m	Mark is dependent on one or more M marks and is for method
A	Mark is dependent on M or m marks and is for accuracy
B	Mark is independent of M or m marks and is for method and accuracy
E	Mark is for explanation
√ **or ft**	Follow through from previous incorrect result
CAO	Correct answer only
CSO	Correct solution only
AWFW	Anything which falls within
AWRT	Anything which rounds to
ACF	Any correct form
AG	Answer given
SC	Special case
oe	Or equivalent
A2, 1	2 or 1 (or 0) accuracy marks
–*x* EE	Deduct *x* marks for each error
NMS	No method shown
PI	Possibly implied
SCA	Substantially correct approach
sf	Significant figure(s)
dp	Decimal place(s)

Q	Answer	Marks	Comments
1	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$	B2,1,0	If not **B2** then award **B1** for either [**N**=] $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or [**M**=] $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **B2** cannot be awarded if the correct matrix for **NM** is obtained by using an incorrect matrix for **N** or **M** or if found by calculating **MN**
		2	

	Question 1 Total	**2**	

Q	Answer	Marks	Comments
2(a)	$\alpha+\beta+\gamma=4$ $\alpha\beta+\beta\gamma+\gamma\alpha=3$	B1	For either $\alpha+\beta+\gamma=4$ or $\alpha\beta+\beta\gamma+\gamma\alpha=3$ seen or used
	$\alpha^2+\beta^2+\gamma^2$ $=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ $=4^2-2(3)$	M1	Correct formula seen/used
	$\alpha^2+\beta^2+\gamma^2=16-6=10$	A1	**CSO AG** Must be convincingly shown
		3	

Q	Answer	Marks	Comments
2(b)	β is a root of the [cubic] equation	E1	**oe**
		1	

Q	Answer	Marks	Comments
2(c)	$\alpha^3+\beta^3+\gamma^3$ $=4(\alpha^2+\beta^2+\gamma^2)-3(\alpha+\beta+\gamma)-3c$	M1	**oe** in terms of c and expressions whose values have been stated/found eg $\alpha^3+\beta^3+\gamma^3=(\alpha+\beta+\gamma)^3$ $-3(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\alpha\gamma)+3(-c)$
	$=4(10)-3(4)-3c=40-12-3c=28-3c$	A1	**AG** Must be convincingly shown
		2	

	Question 2 Total	**6**	

Q	Answer	Marks	Comments
3	$10 = \det(\mathbf{A})\det(\mathbf{B})$ $\det(\mathbf{A})\det(\mathbf{A}^{-1}) = 1$; $5\det(\mathbf{A}) = 1$	**M1**	Either $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ seen/used or $\det(\mathbf{A})\det(\mathbf{A}^{-1}) = 1$ seen/used
	$\det(\mathbf{A}) = \frac{1}{5}$	**A1**	Correct value for $\det(\mathbf{A})$ seen/used **PI** by use of $\det(\mathbf{A})\det(\mathbf{A}^{-1}) = 1$
	$\det(\mathbf{B}) = 50$	**A1ft**	**ft** $10 \div$ their value for $\det(\mathbf{A})$
	Volume of $S_2 = 6 \times 50 = 300$ [cm^3]	**B1ft**	**ft** $6 \times$ their $\lvert\det(\mathbf{B})\rvert$
		4	

	Question 3 Total	**4**	

Q	Answer	Marks	Comments
4(a)	$y = x\sqrt{x} - \frac{1}{3}\sqrt{x} \quad \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}\sqrt{x} - \frac{1}{6\sqrt{x}}$	M1	**ACF** At least one correct term for $\frac{\mathrm{d}y}{\mathrm{d}x}$
	$1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = 1+\frac{9}{4}x - \frac{1}{2} + \frac{1}{36x}$ $= \left(\frac{3}{2}\sqrt{x} + \frac{1}{6\sqrt{x}}\right)^2 \quad = \left(\frac{3}{2}x^{\frac{1}{2}} + \frac{1}{6}x^{-\frac{1}{2}}\right)^2$	A1	A correct expression for $1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ in the form $\left(px^n + qx^{-n}\right)^2$
		2	

Q	Answer	Marks	Comments
4(b)	$S = 2\pi\int_{[1]}^{[4]} y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x$ $= 2\pi\int_{[1]}^{[4]}\left(x\sqrt{x} - \frac{1}{3}\sqrt{x}\right)\left(\frac{3}{2}\sqrt{x} + \frac{1}{6\sqrt{x}}\right)[\mathrm{d}x]$	M1	Substitutes into a correct formula for the surface area S; **ft** their derivative. Condone missing brackets
	$= 2\pi\int_1^4\left(\frac{3}{2}x^2 - \frac{1}{3}x - \frac{1}{18}\right)[\mathrm{d}x]$	M1	Integrand $ax^2 + bx + c$ with at least one correct coefficient
	$= 2\pi\left[\frac{x^3}{2} - \frac{x^2}{6} - \frac{x}{18}\right]_1^4$	A1	
	$= 2\pi\left[\left(\frac{64}{2} - \frac{16}{6} - \frac{4}{18}\right) - \left(\frac{1}{2} - \frac{1}{6} - \frac{1}{18}\right)\right]$ $= 2\pi\left(\frac{63}{2} - \frac{15}{6} - \frac{3}{18}\right) = \pi\left(63 - \frac{16}{3}\right) = \frac{173}{3}\pi$	A1	**AG** Must be convincingly shown
		4	

	Question 4 Total	**6**	

Q	Answer	Marks	Comments
5	$\int 9x^2 \ln x \, dx = 3x^3 \ln x - \int 3x^3 \left(\frac{1}{x}\right) dx$	M1	$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$ $\frac{dv}{dx} = 9x^2 \Rightarrow v = 3x^3$ **PI**
	$\int 9x^2 \ln x \, dx = 3x^3 \ln x - x^3 \quad [+c]$	A1	Correct integration of $9x^2 \ln x$
	$\int_0^e 9x^2 \ln x \, dx = \lim_{a \to 0} \int_a^e 9x^2 \ln x \, dx$ $= \left(3e^3 - e^3\right) - \lim_{a \to 0}\left(3a^3 \ln a - a^3\right)$	M1	Evidence of limit 0 having been replaced by a (**oe**) at any stage and $\lim_{a \to 0}$ seen or taken at any stage with no remaining lim relating to e
	$\lim_{a \to 0}\left(a^3 \ln a\right) = 0$	B1	Accept if stated in the more general format.
	$\int \frac{4}{1+4x^2} dx = \int \frac{1}{\frac{1}{4}+x^2} dx$ $= 2\tan^{-1}(2x) \quad [+c]$	M1 A1	**M1**: $k\tan^{-1}(kx)$ or $\lambda\tan^{-1}(kx)$ **A1**: $2\tan^{-1}(2x)$
	$\int_0^e \left(9x^2 \ln x + \frac{4}{1+4x^2}\right) dx$ $= 2e^3 + 2\tan^{-1}(2e)$	A1	$2e^3 + 2\tan^{-1}(2e)$ Must have scored all previous **M** and **A** marks and no errors when substituting limits.
		7	

	Question 5 Total	7	

Q	Answer	Marks	Comments
6	I.F. is $e^{\int\left(8x\left(x^2+2\right)^{-1}\right)\,dx}=e^{4\ln\left(x^2+2\right)}$	M1	I.F. identified and integration attempted
	I.F. $=\left(x^2+2\right)^4$	A1	Correct integrating factor
	$\left(x^2+2\right)^4\dfrac{dy}{dx}+8x\left(x^2+2\right)^3 y$ $=2x^3\left(x^2+2\right)^4+\dfrac{\left(x^2+2\right)^4}{\left(x^2+2\right)^{\frac{9}{2}}}$		
	$\left(x^2+2\right)^4 y=\int\left(2x^3\left(x^2+2\right)^4+\dfrac{1}{\sqrt{x^2+2}}\right)dx$	M1	Multiplying both sides of the given DE by their I.F. and integrating LHS to get $y\times$ I.F.
	Let $u=x^2+2$ $\Rightarrow\int 2x^3\left(x^2+2\right)^4dx=\int(u-2)u^4du$	M1	A relevant substitution or relevant integration by parts used to find an expression for the integral of $2x^3\left(x^2+2\right)^4$ **PI** by correct integration
	$\int 2x^3\left(x^2+2\right)^4dx=\dfrac{\left(x^2+2\right)^6}{6}-\dfrac{2\left(x^2+2\right)^5}{5}[+A]$	A1	**oe** Correct integration of $2x^3\left(x^2+2\right)^4$ eg $\dfrac{x^2\left(x^2+2\right)^5}{5}-\dfrac{\left(x^2+2\right)^6}{30}[+k]$ or $\dfrac{1}{6}x^{12}+\dfrac{8}{5}x^{10}+6x^8+\dfrac{32}{3}x^6+8x^4[+c]$
	$\int\left(\dfrac{1}{\sqrt{x^2+2}}\right)dx=\sinh^{-1}\left(\dfrac{x}{\sqrt{2}}\right)[+B]$	B1	Correct integration of $\dfrac{1}{\sqrt{x^2+2}}$ **oe**, such as $\ln\left(x+\sqrt{x^2+2}\right)[+B]$
	$y=\dfrac{\left(x^2+2\right)^2}{6}-\dfrac{2\left(x^2+2\right)}{5}+\dfrac{\sinh^{-1}\left(\dfrac{x}{\sqrt{2}}\right)+c}{\left(x^2+2\right)^4}$	A1	Correct GS $y=f(x)$ with **ACF** for f(x)
		7	

	Question 6 Total	7	

Q	Answer	Marks	Comments
7(a)	$\sum_{r=1}^{n}(u_{r+1}-u_r)=\sum_{r=1}^{n}(2u_r)+\sum_{r=1}^{n}4$ LHS = $u_2-u_1+u_3-u_2+u_4-u_3+\ldots+u_n-u_{n-1}+u_{n+1}-u_n$ $=u_{n+1}-u_1$	M1	Uses method of differences with $\sum_{r=1}^{n}(u_{r+1}-u_r)=\sum_{r=1}^{n}(2u_r)+\sum_{r=1}^{n}4$
	RHS $=\sum_{r=1}^{n}(2u_r)+4n$	B1	$\sum_{r=1}^{n}4=4n$ or $\sum_{r=1}^{n}2=2n$
	$2\sum_{r=1}^{n}(u_r)+4n=u_{n+1}-u_1$; $2\sum_{r=1}^{n}(u_r)=u_{n+1}-4n-3$ $\Rightarrow\sum_{r=1}^{n}u_r=\frac{1}{2}u_{n+1}-2n-\frac{3}{2}$	A1	**AG** Must be convincingly shown
		3	

Q	Answer	Marks	Comments
7(b)	When $n=1$, $u_1=5\times3^0-2=5\times1-2=5-2=3$ [Formula is true for $n=1$]	B1	Correct values to show formula true for $n=1$
	Assume formula true for $n=k$ **(*)** , [integer $k\geq1$,] so $u_{k+1}=3\left(5\times3^{k-1}-2\right)+4$	M1	Assumes formula true for $n=k$ and considers $u_{k+1}=3\left(5\times3^{k-1}-2\right)+4$ **oe**
	$u_{k+1}=5\times3^k-6+4$; $u_{k+1}=5\times3^{(k+1)-1}-2$	A1	Be convinced
	Hence formula is true for $n=k+1$ **(**)** and since true for $n=1$ **(***)**, formula $u_n=5\times3^{n-1}-2$ is true for $n=1, 2, 3, \ldots$ by induction (****)	E1	Must have **(*)** , **(**)**, **(***)**, present, previous 3 marks scored and a final statement **(****)** clearly indicating that it relates to positive integers and 'induction'
		4	

Q	Answer	Marks	Comments
7(c)	$\sum_{r=1}^{n}u_r=\frac{1}{2}\left(5\times3^n-2\right)-2n-\frac{3}{2}$ $=\frac{5}{2}\times\left(3^n-1\right)-2n$	B1	**ACF** Accept unsimplified
		1	

	Question 7 Total	**8**	

Q	Answer	Marks	Comments
8(a)	Aux. equation $m^2+2m=0$; $m=-2,\ 0$	M1	Forming and solving the correct aux. equation. **PI** by correct values of m seen/used
	$y=A+B\,e^{-2x}$	A1	Correct general solution
		2	

Q	Answer	Marks	Comments
8(b)	$[\,y_{CF}=A+B\,e^{-2x}\,]$; $y_{PI}=a\,x\,e^{-2x}$	M1	$y_{PI}=a\,x\,e^{-2x}$ seen or used
	$y'_{PI}=a\,e^{-2x}-2a\,x\,e^{-2x}$ $y''_{PI}=-4a\,e^{-2x}+4a\,x\,e^{-2x}$	M1	y'_{PI} and y''_{PI} both of the form $\pm c\,e^{-2x}\pm d\,x\,e^{-2x}$
	$-4ae^{-2x}+4axe^{-2x}+2ae^{-2x}-4ax\,e^{-2x}=6e^{-2x}$ $\Rightarrow a=-3$	M1	Substitution into the DE to form an equation in x and solve to find a value for a
	$[y_{PI}=]-3\,x\,e^{-2x}$	A1	
	$[y_{GS}=]\ A+B\,e^{-2x}-3\,x\,e^{-2x}$	B1ft	Their CF + their PI with exactly two arbitrary constants
	$A+B=0$; $4B+12=4$ (or $-2B-3=1$)		
	$A=2$ and $B=-2$	A1	$A=2$ and $B=-2$
	$y=2-2\,e^{-2x}-3\,x\,e^{-2x}$ When $x=3$ $y=2-11e^{-6}$	A1	$2-11e^{-6}$ **oe**
		7	

	Question 8 Total	**9**	

Q	Answer	Marks	Comments
9(a)	det **A** $= 2k + 12$	B1	Correct expression for det **A**
	Cofactor matrix $\begin{bmatrix} 1 & -3 & 4 \\ 3k+12 & -3k & 2k-12 \\ -4k-17 & 4k+3 & 16-2k \end{bmatrix}$	M1	One complete row or column or diagonal correct
		A2	All nine entries correct else **A1** for at least six entries correct
	Inverse matrix $\mathbf{A}^{-1} =$	M1	Transpose of their cofactors with no more than one further error **and** division by their det **A** $(\neq 0)$
	$= \dfrac{1}{2k+12}\begin{bmatrix} 1 & 3k+12 & -4k-17 \\ -3 & -3k & 4k+3 \\ 4 & 2k-12 & 16-2k \end{bmatrix}$	A1	Correct $\mathbf{A}^{-1}$ scores 6 marks
		6	

Q	Answer	Marks	Comments
9(b)(i)	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$	M1	$\mathbf{A}^{-1}$ **v** for their $\mathbf{A}^{-1}$ with at least one ft component correct.
	$= \dfrac{1}{2k+12}\begin{bmatrix} 1+3k+12-4k-17 \\ -3-3k+4k+3 \\ 4+2k-12+16-2k \end{bmatrix}$	A1ft	At least two **ft** components correct.
	$x = -\left(\dfrac{k+4}{2k+12}\right) \quad y = \dfrac{k}{2k+12} \quad z = \dfrac{8}{2k+12}$	A1	**ACF** All three correct
		3	

Q	Answer	Marks	Comments
9(b)(ii)	$0 < x + y + z < \dfrac{1}{3}$	B1	
		1	

	Question 9 Total	**10**	

Q	Answer	Marks	Comments
10(a)(i)	$\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}+\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}=2\cos\left(\frac{\theta}{2}\right)$	B1	
		1	

Q	Answer	Marks	Comments
10(a)(ii)	$\frac{1}{\mathrm{e}^{\mathrm{i}\theta}+1}=\frac{\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}}{\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}+\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}}=$	M1	**Either** $\frac{1}{\mathrm{e}^{\mathrm{i}\theta}+1}=\frac{\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}}{\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}+\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}}$ **or** $\mathrm{i}\tan\left(\frac{\theta}{2}\right)=\left(\frac{\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}-\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}}{\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}+\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}}\right)$ seen/used $\frac{1}{2}-\frac{\mathrm{i}}{2}\tan\left(\frac{\theta}{2}\right)=\frac{1}{2}-\frac{1}{2}\left[\frac{\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}-\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}}{\mathrm{e}^{\frac{\mathrm{i}\theta}{2}}+\mathrm{e}^{-\frac{\mathrm{i}\theta}{2}}}\right]$
	$=\frac{\cos\left(\frac{\theta}{2}\right)-\mathrm{i}\sin\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{\theta}{2}\right)}=\frac{1}{2}-\frac{\mathrm{i}}{2}\tan\left(\frac{\theta}{2}\right)$	A1	**AG** Must be convincingly shown
		2	

Q	Answer	Marks	Comments
10(b)	$\frac{1}{\mathrm{e}^{\mathrm{i}(\pi-\theta)}+1}=\frac{1}{2}-\frac{\mathrm{i}}{2}\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=\frac{1}{2}-\frac{\mathrm{i}}{2}\cot\left(\frac{\theta}{2}\right)$	M1	
	$\frac{1}{-\mathrm{e}^{-\mathrm{i}\theta}+1}=\frac{1}{2}-\frac{\mathrm{i}}{2}\cot\left(\frac{\theta}{2}\right)$	B1	$\mathrm{e}^{\mathrm{i}\pi}=-1$ seen or used at any stage
	$\frac{1}{-\mathrm{e}^{-\mathrm{i}\theta}+1}-1=-\frac{1}{2}-\frac{\mathrm{i}}{2}\cot\left(\frac{\theta}{2}\right)$ $\frac{\mathrm{e}^{-\mathrm{i}\theta}}{1-\mathrm{e}^{-\mathrm{i}\theta}}=-\frac{1}{2}-\frac{\mathrm{i}}{2}\cot\left(\frac{\theta}{2}\right)$ $\frac{1}{\mathrm{e}^{\mathrm{i}\theta}-1}=-\frac{1}{2}-\frac{\mathrm{i}}{2}\cot\left(\frac{\theta}{2}\right)$	A1	**AG** Must be convincingly shown
		3	

Q	Answer	Marks	Comments
10(c)	$\left(\frac{1}{e^{i\theta}+1}\right)\left(\frac{1}{e^{i\theta}-1}\right)=\frac{1}{e^{2i\theta}-1}$ $=\frac{1}{\cos 2\theta+i\sin 2\theta-1}$ $\left(\frac{1}{e^{i\theta}+1}\right)\left(\frac{1}{e^{i\theta}-1}\right)=$ $\left(\frac{1}{2}-\frac{i}{2}\tan\left(\frac{\theta}{2}\right)\right)\left(-\frac{1}{2}-\frac{i}{2}\cot\left(\frac{\theta}{2}\right)\right)$ $=-\frac{1}{2}+\frac{i}{4}\left(\tan\left(\frac{\theta}{2}\right)-\cot\left(\frac{\theta}{2}\right)\right)$	M1	Considers a relevant combination of $\frac{1}{e^{i\theta}+1}$ and $\frac{1}{e^{i\theta}-1}$ to obtain either $\frac{1}{\cos 2\theta+i\sin 2\theta-1}$ or $a+i\,b\left(\tan\left(\frac{\theta}{2}\right)-\cot\left(\frac{\theta}{2}\right)\right)$ with a or b correct **oe** eg $\frac{1}{2}\left(\frac{1}{e^{i\theta}-1}-\frac{1}{e^{i\theta}+1}\right)=\frac{1}{e^{2i\theta}-1}$ $=\frac{1}{\cos 2\theta+i\sin 2\theta-1}$ $\frac{1}{2}\left(\frac{1}{e^{i\theta}-1}-\frac{1}{e^{i\theta}+1}\right)=$ $\frac{1}{2}\left\{-\frac{1}{2}-\frac{i}{2}\cot\left(\frac{\theta}{2}\right)-\left(\frac{1}{2}-\frac{i}{2}\tan\left(\frac{\theta}{2}\right)\right)\right\}$ $=-\frac{1}{2}+\frac{i}{4}\left(\tan\left(\frac{\theta}{2}\right)-\cot\left(\frac{\theta}{2}\right)\right)$
	$\frac{1}{\cos 2\theta-1+i\sin 2\theta}$ $=-\frac{1}{2}+i\frac{1}{4}\left(\tan\left(\frac{\theta}{2}\right)-\cot\left(\frac{\theta}{2}\right)\right)$	A1	$\frac{1}{\cos 2\theta-1+i\sin 2\theta}$ $=-\frac{1}{2}+i\frac{1}{4}\left(\tan\left(\frac{\theta}{2}\right)-\cot\left(\frac{\theta}{2}\right)\right)$
		2	

	Question 9 Total	**8**	

Q	Answer	Marks	Comments
11(a)	$\begin{bmatrix}-2\\1\\2\end{bmatrix} \times \begin{bmatrix}1\\-3\\4\end{bmatrix} = \begin{bmatrix}10\\10\\5\end{bmatrix}$	B1	$\begin{bmatrix}10\\10\\5\end{bmatrix}$ **oe**
		1	

Q	Answer	Marks	Comments
11(b)	$\left(\begin{bmatrix}-2\\1\\2\end{bmatrix} \times \begin{bmatrix}1\\-3\\4\end{bmatrix}\right) \bullet \begin{bmatrix}-4\\3\\-2\end{bmatrix} = \begin{bmatrix}10\\10\\5\end{bmatrix} \bullet \begin{bmatrix}-4\\3\\-2\end{bmatrix} =$	M1	Correct method to evaluate a relevant scalar triple product; ft their **(a)** Condone a miscopy of sign from given vectors
	$= -40 + 30 - 10\ [= -20] \neq 0$ so they are not coplanar vectors	A1ft	Must see correct ft evaluation and explicit comparison with 0, or relevant initial statement involving 0, followed by a correct ft conclusion.
		2	

Q	Answer	Marks	Comments
11(c)	Direction vector for L: $\mathbf{d} = \begin{bmatrix}2\\3\\-6\end{bmatrix}$ Direction of normal to the plane: $\mathbf{n} = \begin{bmatrix}2\\2\\1\end{bmatrix}$	B1ft	Seen or used; **d** correct and **ft** their answer to part (a) for **n** in a simplified or unsimplified form
	$\mathbf{n} \,.\, \mathbf{d} = 4 + 6 - 6$	M1	Finds a numerical expression for the scalar product of their normal to Π and their **d** **PI**
	$\sqrt{2^2 + 2^2 + 1^2}\sqrt{2^2 + 3^2 + (-6)^2}\cos\theta = 4$	m1	**ft** their **n** and **d** **PI**
	$\cos\theta = \dfrac{4}{21}$ Angle between L and $\Pi = 90^\circ - \cos^{-1}\left(\dfrac{4}{21}\right)$		
	$= 11^\circ$ to nearest degree	A1	**CAO** 11° (Condone missing $^\circ$)
		4	

Q	Answer	Marks	Comments
11(d)	Line L: $\mathbf{r} = \begin{bmatrix}1\\0\\2\end{bmatrix} + t\begin{bmatrix}2\\3\\-6\end{bmatrix}$		
	General point on L is $(1+2t, 3t, 2-6t)$	M1	Finding a general point on L
	$1+2t=1-2\lambda+\mu$ $3t=-2+\lambda-3\mu$; $\quad 8t=-4-5\mu$ $2-6t=3+2\lambda+4\mu$; $\quad 3-4t=4+5\mu$ $t=-\frac{3}{4}$	M1	Eliminating **r** for L and Π, forming and solving three equations in three unknowns (or substituting general pt on L into $2x+2y+z=1$) to find a value for t
	$P\left(-\frac{1}{2}, -\frac{9}{4}, \frac{13}{2}\right)$	A1	Correct coordinates or position vector for P **PI** or $\left\lvert -\frac{3}{4}\right\rvert \times \sqrt{2^2+3^2+(-6)^2}$
	$PQ=\sqrt{1.5^2+2.25^2+4.5^2}=5.25$	A1	Correct distance for PQ
		4	

Q	Answer	Marks	Comments
11(d) ALT	Plane Π: $\mathbf{r} = \begin{bmatrix}1-2\lambda+\mu\\-2+\lambda-3\mu\\3+2\lambda+4\mu\end{bmatrix}$		
	Sub in L: $\begin{bmatrix}-2\lambda+\mu\\-2+\lambda-3\mu\\1+2\lambda+4\mu\end{bmatrix} \times \begin{bmatrix}2\\3\\-6\end{bmatrix} = \mathbf{0}$		
	$9-12\lambda+6\mu=0$ $2-8\lambda+14\mu=0$ $4-8\lambda+9\mu=0$	M1	Eliminating **r** for L and Π, forming at least two equations in two unknowns
	$\mu=\frac{2}{5}$ $\quad \lambda=\frac{19}{20}$	M1	Solving three correct equations to obtain a value for each of the unknowns
	$P\left(-\frac{1}{2}, -\frac{9}{4}, \frac{13}{2}\right)$	A1	Correct coordinates or position vector for P **PI** or $\left\lvert -\frac{3}{4}\right\rvert \times \sqrt{2^2+3^2+(-6)^2}$
	$PQ=\sqrt{1.5^2+2.25^2+4.5^2}=5.25$	A1	Correct distance for PQ
		4	

Q	Answer	Marks	Comments
11(e)	Let *T* be the point on Π such that *QT* is perpendicular to Π. *Q* is a point on line *L* so angle *PQT*, is the angle θ from **part (c)** between normal to Π and line *L*. $QT = PQ\cos\theta$	M2	**oe** eg $QT = PQ\sin\left(90^\circ - \theta\right)$ condoning their rounded answer for $90^\circ - \theta$ found in **part (c)**
	Shortest distance $= 5.25 \times \dfrac{4}{21} = 1$	A1	**CAO** Do not accept 1 from non-exact values
		3	

Q	Answer	Marks	Comments
11(e) ALT	$1 + 2t = 1 - 2\lambda + \mu$ $2t = -2 + \lambda - 3\mu$; $6t = -4 - 5\mu$ $2 + t = 3 + 2\lambda + 4\mu$; $3 + 3t = 4 + 5\mu$ $t = -\dfrac{1}{3}$	M1	Finds a general point on *QT* and eliminates **r** for *QT* and Π, forming and solving three equations in three unknowns **oe** to find a value for t
	$T\left(\dfrac{1}{3}, -\dfrac{2}{3}, \dfrac{5}{3}\right)$ $Q(1, 0, 2)$ $QT = \sqrt{\left(\dfrac{2}{3}\right)^2 + \left(\dfrac{2}{3}\right)^2 + \left(\dfrac{1}{3}\right)^2}$	m1	Finds coordinates of *T* and uses distance formula to find a value for the distance *QT* **oe**
	Shortest distance $= 1$	A1	**CAO** Must be from exact values.
		3	

	Question 11 Total	**14**	

Q	Answer	Marks	Comments
12(a)	$y = e^{\frac{7}{25}x}(\cosh x)^{-1}$, $\frac{dy}{dx} = \frac{7}{25}e^{\frac{7}{25}x}(\cosh x)^{-1} - e^{\frac{7}{25}x}(\cosh x)^{-2}\sinh x$	**M1** **A1**	Use of the product formula or quotient formula. **ACF**
	At *P*, $\frac{dy}{dx} = 0 \Rightarrow \frac{7}{25}\cosh x - \sinh x = 0$	**M1**	
	$\tanh x = \frac{7}{25}$	**A1**	
	$x = \tanh^{-1}\left(\frac{7}{25}\right) = \frac{1}{2}\ln\left(\frac{1+\frac{7}{25}}{1-\frac{7}{25}}\right)$	**M1**	Using $\tanh^{-1}(n) = \frac{1}{2}\ln\left(\frac{1+n}{1-n}\right)$
	$x = \ln\left(\frac{4}{3}\right)$	**A1**	$x = \ln\left(\frac{4}{3}\right)$
		6	

Q	Answer	Marks	Comments
12(b)	$y_P = e^{\frac{7}{25}(\ln k)}\,\text{sech}(\ln k)$	**M1**	Attempts to find the y-coordinate of *P*, **ft** their value of k from **part (a)**
	$\text{sech}(\ln k) = \frac{2k}{k^2+1}$; $e^{\frac{7}{25}(\ln k)} = k^{\frac{7}{25}}$	**B1ft**	$e^{\frac{7}{25}(\ln k)} = k^{\frac{7}{25}}$ and $\text{sech}(\ln k) = \frac{2k}{k^2+1}$ **oe** **ft** their k value
	Shortest distance = $\left(\frac{24}{25}\right)\left(\frac{4}{3}\right)^{\frac{7}{25}}$	**A1**	**oe** but must be in the required printed form eg $\left(\frac{18}{25}\right)\left(\frac{4}{3}\right)^{\frac{32}{25}}$
		3	

Q	Answer	Marks	Comments
12(c)	Shortest distance $\left(\frac{24}{25}\right)\left(\frac{4}{3}\right)^{\frac{7}{25}} = 1.04\ldots$	**B1**	1.04…

	Since $[-1<]\tanh x < 1$, and line *L*,
	 $y = 1.04\ldots$ is above $y = 1$, *L* does not intersect the curve $y = \tanh x$
		E1ft	States $\tanh x < 1$ and
			 if (their shortest distance) < 1 states *L* intersects $y = \tanh x$ **oe**
			 if (their shortest distance) ≥ 1 states *L* does not intersect $y = \tanh x$ **oe**

		2	

	Question 12 Total	**11**	

Q	Answer	Marks	Comments
13(a)	$\left[\ln(1+4x)=\right] \quad 4x-8x^2+\frac{64}{3}x^3$	B1	Ignore higher order terms
	[valid for] $-\frac{1}{4}<x\le\frac{1}{4}$	B1	
		2	

Q	Answer	Marks	Comments
13(b)(i)	$y=\ln(\cos x-\sin x)$;		
	$\frac{dy}{dx}=\frac{-\sin x-\cos x}{\cos x-\sin x}=1-\frac{2\cos x}{\cos x-\sin x}$;	B1	**ACF** for the first derivative
	$\frac{d^2y}{dx^2}=$ $\frac{2\sin x(\cos x-\sin x)+2\cos x(-\sin x-\cos x)}{(\cos x-\sin x)^2}$	M1	Quotient rule used
	$\frac{d^2y}{dx^2}=\frac{-2(\sin^2x+\cos^2x)}{\cos^2x+\sin^2x-2\sin x\cos x}$		
	$\frac{d^2y}{dx^2}=\frac{-2}{1-\sin 2x}$	A1	**AG** Must be convincingly shown
		3	

Q	Answer	Marks	Comments
13(b)(ii)	$\frac{d^3y}{dx^3}=\frac{0+2(-2\cos 2x)}{(1-\sin 2x)^2}$	B1	**ACF** for the third derivative
	$y(0)=0; y'(0)=-1; y''(0)=-2; y'''(0)=-4$	M1	All 4 attempted with at least 2 correct
	$\ln(\cos x-\sin x)=0-1x+\frac{(-2)x^2}{2!}+\frac{(-4)x^3}{3!}$		
	$\ln(\cos x-\sin x)=-x-x^2-\frac{2}{3}x^3$	A1	**AG** Must be convincingly shown
		3	

Q	Answer	Marks	Comments
13(c)	$\ln\left[(1-\sin 2x)\sqrt{1+4x}\right] =$ $= \ln(1-\sin 2x) + \ln\sqrt{1+4x}$	M1	Seen or used
	$= 2\ln(\cos x - \sin x) + \frac{1}{2}\ln(1+4x)$	M1	$\ln(1-\sin 2x) = 2\ln(\cos x - \sin x)$ seen or used
	$= 2\left(-x - x^2 - \frac{2}{3}x^3\right) + \frac{1}{2}\left(4x - 8x^2 + \frac{64}{3}x^3\right)$	A1ft	**ft** their 3-term expansion in **part (a)**
	$\lim_{x\to 0}\left[\frac{\ln\left((1-\sin 2x)\sqrt{1+4x}\right)}{5x^2+6x^3}\right]$ $= \lim_{x\to 0}\left[\frac{-6x^2 + \frac{28}{3}x^3 \ldots}{5x^2+6x^3}\right]$ $= \lim_{x\to 0}\left[\frac{-6 + \frac{28}{3}x \ldots}{5+6x}\right]$ [so the limit exists]	M1	Substitutes series expansions and divides numerator and denominator by x^2 to reach the form $\lim_{x\to 0}\left[\frac{P+O(x)}{Q+O(x)}\right]$, so limit exists $= \frac{P}{Q}$
	$\lim_{x\to 0}\left[\frac{\ln\left((1-\sin 2x)\sqrt{1+4x}\right)}{5x^2+6x^3}\right] = -\frac{6}{5}$	A1	**CSO**
		5	

	Question 13 Total	**13**	

Q	Answer	Marks	Comments
14(a)	$B\left(6+3\sqrt{2}\ ,\ \frac{7\pi}{4}\right)$; $OA = OB = 6+3\sqrt{2}$ Angle $AOB = \frac{\pi}{4}+\frac{\pi}{4}=\frac{\pi}{2}$	M1	Uses polar coordinates of A and B **oe** to find two relevant lengths and an angle to use in finding the required area **PI**
	Area of triangle AOB = $\frac{1}{2}\left(6+3\sqrt{2}\right)^2=\frac{1}{2}\left(54+36\sqrt{2}\right)=27+18\sqrt{2}$	A1	**AG** Must be convincingly shown
		2	

Q	Answer	Marks	Comments
14(b)	$r=\frac{3}{2}\text{cosec}^2\left(\frac{\theta}{2}\right)$; $r=\frac{3}{1-\cos\theta}$	M1	$r=\frac{3}{1-\cos\theta}$ condone sign error
	$r-r\cos\theta=3$; $r=3+x$	M1	Use of $r\cos\theta=x$ to eliminate θ
	$r^2=(3+x)^2$; $x^2+y^2=(3+x)^2$	M1	$r^2=x^2+y^2$ used at any stage
	$y^2=6x+9$	A1	$y^2=6x+9$ **oe** for $f(x)$
		4	

Q	Answer	Marks	Comments
14(c)(i)	$\tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}, \quad \theta = \frac{4\pi}{3}$		
	When $\theta = \frac{\pi}{3}$, $r = 6$ When $\theta = \frac{4\pi}{3}$, $r = 2$	**B2,1,0**	**B2** any three correct; else **B1** for any two correct
	Polar coordinates of *P* and *Q* are $\left(6, \frac{\pi}{3}\right)$, $\left(2, \frac{4\pi}{3}\right)$	**B1**	$\left(6, \frac{\pi}{3}\right)$, $\left(2, \frac{4\pi}{3}\right)$
		3	

Q	Answer	Marks	Comments
14(c)(ii)	$\int \left(1 + \cot^2\left(\frac{\theta}{2}\right)\right) \operatorname{cosec}^2\left(\frac{\theta}{2}\right) \mathrm{d}\theta$ Let $u = \cot\left(\frac{\theta}{2}\right)$, $\frac{\mathrm{d}u}{\mathrm{d}\theta} = -\frac{1}{2}\operatorname{cosec}^2\left(\frac{\theta}{2}\right)$ $\int \operatorname{cosec}^4\left(\frac{\theta}{2}\right) \mathrm{d}\theta = \int \left(1 + u^2\right)(-2)\ \mathrm{d}u$	**M1**	Uses a relevant substitution or integration by parts so as to require a single step to find the integral.
	$= -2\cot\left(\frac{\theta}{2}\right) - \frac{2}{3}\cot^3\left(\frac{\theta}{2}\right) \quad [+c]$	**A2,1,0**	**PI** by the **A1** form below If not **A2**, award **A1** for the form $k\left(\cot\left(\frac{\theta}{2}\right) + \frac{1}{3}\cot^3\left(\frac{\theta}{2}\right)\right) \quad [+c]$ where $k = 2$ or $\frac{1}{2}$ or $-\frac{1}{2}$
		3	

Q	Answer	Marks	Comments
14(c)(iii)	Area $= \frac{1}{2}\int_{\left[\frac{\pi}{3}\right]}^{\left[\frac{4\pi}{3}\right]} \frac{9}{4}\text{cosec}^4\left(\frac{\theta}{2}\right)[\text{d}\theta]$	M1	Use of $\frac{1}{2}\int r^2\,[\text{d}\theta]$ oe with integration attempted. Condone missing or incorrect limits
	$= \frac{9}{8}\left[-2\cot\left(\frac{\theta}{2}\right) - \frac{2}{3}\cot^3\left(\frac{\theta}{2}\right)\right]_{\frac{\pi}{3}}^{\frac{4\pi}{3}}$		
	$= \frac{9}{8}\left(\left\{-2\cot\left(\frac{2\pi}{3}\right) - \frac{2}{3}\cot^3\left(\frac{2\pi}{3}\right)\right\} - \left\{-2\cot\left(\frac{\pi}{6}\right) - \frac{2}{3}\cot^3\left(\frac{\pi}{6}\right)\right\}\right)$	M1	Uses their answer to **part (c)(ii)** and substitutes their non-zero values for θ found in **part (c)(i)** as limits with the appropriate subtraction included. **PI** by the line above in the Answer column followed by the correct final answer.
	$= \frac{9}{8}\left(\left\{-2\left(-\frac{1}{\sqrt{3}}\right) - \frac{2}{3}\left(-\frac{1}{\sqrt{3}}\right)^3\right\} - \left\{-2\left(\sqrt{3}\right) - \frac{2}{3}\left(\sqrt{3}\right)^3\right\}\right)$		
	$= \frac{9}{8}\left(\left\{\frac{2}{\sqrt{3}} + \frac{2}{9\sqrt{3}}\right\} - \left\{-2\sqrt{3} - 2\sqrt{3}\right\}\right)$		
	$= \frac{16}{3}\sqrt{3}$	A1	**CAO**
		3	

	Question 14 Total	**15**	