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Mark scheme

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Key to mark scheme abbreviations

M	Mark is for method
m	Mark is dependent on one or more M marks and is for method
A	Mark is dependent on M or m marks and is for accuracy
B	Mark is independent of M or m marks and is for method and accuracy
E	Mark is for explanation
√ **or ft**	Follow through from previous incorrect result
CAO	Correct answer only
CSO	Correct solution only
AWFW	Anything which falls within
AWRT	Anything which rounds to
ACF	Any correct form
AG	Answer given
SC	Special case
oe	Or equivalent
A2, 1	2 or 1 (or 0) accuracy marks
–*x* EE	Deduct *x* marks for each error
NMS	No method shown
PI	Possibly implied
SCA	Substantially correct approach
sf	Significant figure(s)
dp	Decimal place(s)

Q	Answer	Marks	Comments
1(a)	$\begin{vmatrix} 25-1 & 8 \\ t & 3-1 \end{vmatrix} = 0$	**M1**	**M1** for forming an equation such as $\begin{vmatrix} 25-\lambda & 8 \\ t & 3-\lambda \end{vmatrix} = 0$ with either $\lambda = 1$ **or** $\lambda = 27$ or $\begin{vmatrix} 25 & 8 \\ t & 3 \end{vmatrix} =$ **product** of the eigenvalues or solving simultaneous equations $25x + 8y = \lambda x$ and $tx + 3y = \lambda y$ with either $\lambda = 1$ **or** $\lambda = 27$ to find a value for t
	$t = 6$	**A1**	**CAO** **NMS** 0/2
		2	

Q	Answer	Marks	Comments
1(b)(i)	[invariant lines are] **r** = $\mu\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $y = 0.25x$; **r** = $\mu\begin{bmatrix} -1 \\ 3 \end{bmatrix}$, $y = -3x$	**B1**	**oe**
		1	

Q	Answer	Marks	Comments
1(b)(ii)	line of invariant points is $y = -3x$	**B1ft**	Clearly identifies their line
	since it corresponds to $\lambda = 1$	**E1**	**or** fuller explanation
		2	

	Question 1 Total	**5**	

Q	Answer	Marks	Comments
2	$\int(1+x)e^{-2x}\,dx$; $u=1+x \Rightarrow du=dx$ $dv=e^{-2x}dx \Rightarrow v=-\frac{1}{2}e^{-2x}$	M1	PI $u=1+x$; $dv=e^{-2x}dx$ $du=dx$; $v=-\frac{1}{2}e^{-2x}$ [the choice simplifies the integration]
	$\int(1+x)e^{-2x}\,dx$ $=-\frac{1}{2}e^{-2x}(1+x)+\int\frac{1}{2}e^{-2x}dx$	A1	PI
	$=-\frac{1}{2}e^{-2x}(1+x)-\frac{1}{4}e^{-2x}\ [+c]$	A1	Fully correct integration of $(1+x)e^{-2x}$
	$I=\int_{-1}^{\infty}(1+x)e^{-2x}\,dx$ $=\lim_{a\to\infty}\int_{-1}^{a}(1+x)e^{-2x}\,dx$	M1	Evidence of limit ∞ replaced by a (**oe**) $\lim_{a\to\infty}$ seen or taken at any stage with no remaining $\lim$ relating to -1
	$=\lim_{a\to\infty}\left[-\frac{1}{2}e^{-2a}(1+a)-\frac{1}{4}e^{-2a}-\left(-\frac{1}{4}e^{2}\right)\right]$		
	$\lim_{a\to\infty}\left(ae^{-2a}\right)=0$	B1	Accept if stated in the more general format.
	$I=\frac{1}{4}e^{2}$	A1	**CAO** Must have scored the first 4 marks for this mark to be awarded
		6	

	Question 2 Total	**6**	

Q	Answer	Marks	Comments
3(a)	Det = $3\begin{vmatrix} k & 3 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 5 & 3 \\ k+2 & 2 \end{vmatrix} + 1\begin{vmatrix} 5 & k \\ k+2 & 1 \end{vmatrix}$	M1	**oe** Correctly expanding by any row **or** column
	$= 3(2k-3)+10-3(k+2)+5-k(k+2)$ $= 6k-9+10-3k-6+5-k^2-2k$ $= k-k^2$	A1	**AG** Be convinced (must see correct expansion of the brackets)
		2	

Q	Answer	Marks	Comments
3(b)(i)	[$k=1$, $\triangle = 0$, no unique point]		
	$3x - y + z = 11$ **(1)** $5x + y + 3z = 10$ **(2)** $3x + y + 2z = -2$ **(3)**	B1	Correct system of equations in the case $k = 1$
	(1) + **(2)** ⇒ $8x + 4z = 21$ ⇒ $2x + z = 5.25$	M1	**oe** Eliminating one variable in order to compare two simultaneous equations
	(1) + **(3)** ⇒ $6x + 3z = 9$ ⇒ $2x + z = 3$		
	[Inconsistent so] no solutions	A1	From comparing correct equations. Note: **(2)** – **(3)** ⇒ $2x + z = 12$
		3	

Q	Answer	Marks	Comments
3(b)(ii)	Three planes form a [triangular] prism	E1	**oe**
		1	

	Question 3 Total	**6**	

Q	Answer	Marks	Comments
4	I.F. is $e^{\int \tanh x \, dx} \left[= e^{\ln \cosh x}\right]$	M1	I.F. identified and integration attempted
	$= \cosh x$	A1	Correct integrating factor
	$y \cosh x = \int \cosh^3 x \, dx + \int 2e^x \cosh x \, dx$	m1	Multiplying both sides of the given DE by the I.F. and integrating LHS to get $y \times$I.F.
	$= \int \left(1 + \sinh^2 x\right) d(\sinh x) + \int \left(e^{2x} + 1\right) dx$	M1 M1	Writing each integral in a suitable form for direct integration, **PI** by later work
	$y \cosh x = \sinh x + \frac{1}{3}\sinh^3 x + \frac{1}{2}e^{2x} + x + A$	A2,1,0	**oe** If not **A2**, **A1** can be awarded for either $y \cosh x = \sinh x + \frac{1}{3}\sinh^3 x + ... + A$ **oe** **or** $y \cosh x = ... + \frac{1}{2}e^{2x} + x + A$ **oe**
		7	

	Question 4 Total	**7**	

Q	Answer	Marks	Comments
5(a)(i)	$\beta = 3 + \sqrt{3}\,\mathrm{i}$	B1	
		1	

Q	Answer	Marks	Comments
5(a)(ii)	$\alpha\beta\gamma = -\left(-\frac{12}{4}\right)$	M1	
	$12\gamma = 3 \quad \Rightarrow \quad \gamma = \frac{1}{4}$	A1	oe
		2	

Q	Answer	Marks	Comments
5(a)(iii)	$\alpha + \beta + \gamma = -\left(\frac{c}{4}\right)$; $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{d}{4}$	M1	Either one seen/used **or** **ALT:** Forming two simultaneous equations in c and d by substituting value(s) of root(s) into cubic equation **eg** $6c + 3d - 12=0$; $-d - 6c - 96=0$
	$\frac{25}{4} = -\left(\frac{c}{4}\right) \quad \Rightarrow \quad c = -25$	A1ft	ft on candidate's γ so $c = -4(6+\gamma)$
	$12 + 1.5 = \frac{d}{4} \quad \Rightarrow \quad d = 54$	A1	Correct value for d
		3	

Q	Answer	Marks	Comments
5(b)(i)	$3-\sqrt{3}\,\mathrm{i}=\sqrt{12}\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{6}}$	**B1** **B1**	$r=\sqrt{12}$ **oe** exact value $\theta=-\frac{\pi}{6}$
		2	

Q	Answer	Marks	Comments
5(b)(ii)	$\alpha^n=\left\{\sqrt{12}\left[\cos\left(-\frac{\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{\pi}{6}\right)\right]\right\}^n$	**B1ft**	**ft** on c's values for r and θ **PI** by later work
	$\alpha^n=\left(\sqrt{12}\right)^n\left[\cos\left(-\frac{n\pi}{6}\right)+\mathrm{i}\sin\left(-\frac{n\pi}{6}\right)\right]$	**M1**	**PI** Equivalent to de Moivre for either α^n **or** β^n
	$\beta^n=\left(\sqrt{12}\right)^n\left[\cos\left(\frac{n\pi}{6}\right)+\mathrm{i}\sin\left(\frac{n\pi}{6}\right)\right]$	**A1**	Correct α^n **or** β^n **or** $\alpha^n+\beta^n$ in trigonometric or exponential form
	$\alpha^n+\beta^n=2\left(\sqrt{12}\right)^n\cos\left(\frac{n\pi}{6}\right)$	**A1**	$A\cos\left(\frac{n\pi}{6}\right)$ allowing any correct exact form for A
		4	

Q	Answer	Marks	Comments
5(b)(iii)	$\alpha^n+\beta^n=0\Rightarrow\cos\left(\frac{n\pi}{6}\right)=0$ $\Rightarrow\frac{n\pi}{6}=(2k-1)\frac{\pi}{2}$	**M1**	$\frac{n\pi}{6}=(2k-1)\frac{\pi}{2}$. **oe** Must be using $\alpha^n+\beta^n=k\cos\left(\frac{n\pi}{6}\right)$ ft on candidate's θ from **(b)(i)**
	Since n is a positive integer, $n=3(2k-1)$, integer $k\geq1$	**A1**	$n=3(2k-1)$, integer $k\geq1$ **oe** **eg** 'n = odd positive multiples of 3'
		2	

	Question 5 Total	**14**	

Q	Answer	Marks	Comments
6(a)(i)	$\frac{1}{(r+2)(r+3)}=\frac{A}{r+2}+\frac{B}{r+3}$	M1	**PI** Forming partial fractions and attempt to find A **or** B
	$A=1$; $B=-1$	A1	$A=1$; $B=-1$
	$\sum_{r=1}^{n}\frac{1}{(r+2)(r+3)}=\frac{1}{3}-\frac{1}{4}+\frac{1}{4}-\frac{1}{5}+\ldots$ $\ldots+\frac{1}{n+1}-\frac{1}{n+2}+\frac{1}{n+2}-\frac{1}{n+3}$	M1	Uses method of differences showing at least terms which cancel
	$=\frac{1}{3}-\frac{1}{n+3}$	A1	**AG** Be convinced
		4	

Q	Answer	Marks	Comments
6(a)(ii)	When n =1, LHS = $\frac{2}{24}=\frac{1}{12}$, RHS = $\frac{1}{6}-\frac{1}{12}=\frac{1}{12}$ [so formula is true for n = 1]	B1	Correct values
	Assume formula true for $n=k$ **(*)** , integer $k\geq1$, so $\sum_{r=1}^{k+1}\frac{2}{(r+1)(r+2)(r+3)}=$ $\frac{1}{6}-\frac{1}{(k+2)(k+3)}+\frac{2}{(k+2)(k+3)(k+4)}$	M1	Assumes the result true for $n=k$ and considers $\sum_{r=1}^{k+1}\frac{2}{(r+1)(r+2)(r+3)}$
	$=\frac{1}{6}-\frac{k+4-2}{(k+2)(k+3)(k+4)}$ $=\frac{1}{6}-\frac{1}{(k+3)(k+4)}$	A1	Be convinced
	Hence formula is true for n = k+1 **(**)** and since true for n = 1**(***)**, formula is true for n = 1, 2, 3, … (****) by induction	E1	Must have **(*)** , **(**)** & **(***)** present, previous 3 marks scored and final statement **(****)** clearly indicating that it relates to positive integers
		4	

Q	Answer	Marks	Comments
6(b)	$\sum_{r=1}^{n}\frac{r}{(r+1)(r+2)(r+3)}=\sum_{r=1}^{n}\frac{1}{(r+2)(r+3)}-\sum_{r=1}^{n}\frac{1}{(r+1)(r+2)(r+3)}$	M1	Writes the given summation as a difference so that **(a)** and **(b)** results can be used
	$=\left[\frac{1}{3}-\frac{1}{n+3}\right]-\frac{1}{2}\left[\frac{1}{6}-\frac{1}{(n+2)(n+3)}\right]$	A1	$\left[\frac{1}{3}-\frac{1}{n+3}\right]-\frac{1}{2}\left[\frac{1}{6}-\frac{1}{(n+2)(n+3)}\right]$
	$=\frac{1}{4}+\frac{1-2(n+2)}{2(n+2)(n+3)}$		
	$=\frac{n^2+5n+6+2-4n-8}{4(n+2)(n+3)}$		
	$=\frac{n(n+1)}{4(n+2)(n+3)}$	A1	$\frac{n(n+1)}{4(n+2)(n+3)}$ obtained convincingly
		3	

	Question 6 Total	**11**	

Q	Answer	Marks	Comments
7(a)	$y_{PI} = a x^2 e^{-3x} + b$		
	$y'_{PI} = 2a x e^{-3x} - 3a x^2 e^{-3x}$	M1	Differentiates ax^2e^{-3x} as $\pm pxe^{-3x} \pm qx^2e^{-3x}$ form
	$y''_{PI} = e^{-3x}\left(2a - 12ax + 9ax^2\right)$	A1	y'_{PI} and y''_{PI} both correct
	$e^{-3x}\left(2a - 12ax + 9ax^2 + 12ax - 18ax^2 + 9ax^2\right) + 9b = 9e^{-3x} + 18$ $\Rightarrow 2a = 9$ and $9b = 18$	M1	Substitutes into the given DE, **ft** their derivatives, and equates coefficients to obtain two equations, at least one correct.
	$\Rightarrow a = 4.5$	A1	Correct value for a with no errors seen in any term involving x
	$\Rightarrow b = 2$	B1	$b = 2$
		5	

Q	Answer	Marks	Comments
7(b)	Aux equation $m^2 + 6m + 9 = 0$ $(m+3)^2 = 0 \Rightarrow m = -3$	M1	Factorising **or** using quadratic formula **oe** on correct aux. equation. **PI** by correct value of m seen/used
	$\left[y_{CF} =\right] (Ax + B)e^{-3x}$	A1	Correct CF
	$\left[y_{GS} =\right] (Ax + B)e^{-3x} + 4.5x^2e^{-3x} + 2$	B1ft	(c's CF + c's PI) but must have exactly two arbitrary constants in CF
	$x = 0,\ y = 3 \Rightarrow 3 = B + 2 \Rightarrow B = 1$	A1ft	**Ft** on $B = 3 - c's\ b$
	$x = 0,\ y' = 0 \Rightarrow 0 = A - 3B \Rightarrow A = 3$	A1ft	**Ft** on $A = 3 \times c's\ B$
	$y = \left(3x + 1 + 4.5x^2\right)e^{-3x} + 2$	A1	
		6	
	Question 7 Total	**11**	

Q	Answer	Marks	Comments
8(a)	det **M** $= 6 - 4k$	B1	Seen or used
	Cofactor matrix		
	$\begin{bmatrix} 6 & 2 & 3k+4 \\ -6 & -2 & -k-7 \\ 6-2k & 4-2k & 8-k-k^2 \end{bmatrix}$	M1	One complete row or column correct **PI** by later work
		A2,1,0	**A2** all nine correct; else **A1** at least six correct **PI** by later work
	Inverse matrix $\mathbf{M}^{-1} =$		
	$\frac{1}{6-4k}\begin{bmatrix} 6 & -6 & 6-2k \\ 2 & -2 & 4-2k \\ 3k+4 & -k-7 & 8-k-k^2 \end{bmatrix}$	M1	Transpose of their cofactors with no more than one further error **and** division by their det **M** provided det **M** $\neq 0$ when k is an integer
		A1	**CAO**
		6	

Q	Answer	Marks	Comments
8(b)	$[\mathbf{A}^{-1} =]\ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	B2,1,0	If not **B2**, then **B1** for $\begin{bmatrix} \cos(-90^\circ) & -\sin(-90^\circ) & 0 \\ \sin(-90^\circ) & \cos(-90^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **or** better
		2	

	Question 8 Total	8	

Q	Answer	Marks	Comments
9(a)	$\tan y = \frac{1+x}{1-x}$		
	$\sec^2 y \frac{dy}{dx} = \frac{(1-x)(1)-(1+x)(-1)}{(1-x)^2}$	**M1**	Correct differentiation wrt x of either $\tan y$ or $\frac{1+x}{1-x}$
	$\left(1+\left(\frac{1+x}{1-x}\right)^2\right)\frac{dy}{dx} = \frac{2}{(1-x)^2}$	**m1**	Replacing $\sec^2 y$ by $1+\left(\frac{1+x}{1-x}\right)^2$ Accept if part of the differentiation of $\tan^{-1}\left(\frac{1+x}{1-x}\right)$
	$\frac{dy}{dx} = \frac{2}{(1-x)^2+(1+x)^2} = \frac{1}{1+x^2}$	**A1**	**AG** Be convinced
		3	

Q	Answer	Marks	Comments
9(b)	$\frac{d}{dx}\left(\tan^{-1}\left(\frac{1+x}{1-x}\right)\right) = \frac{1}{1+x^2}$		
	$\tan^{-1}\left(\frac{1+x}{1-x}\right) = \tan^{-1}x\ [+c]$	**M1**	Integrates both sides wrt x **oe** to obtain $\tan^{-1}\left(\frac{1+x}{1-x}\right) = \tan^{-1}x\ [+c]$
	when $x=0$, $\tan^{-1}1 = 0 + c \Rightarrow c = \frac{\pi}{4}$	**m1**	Finds a value of the constant of integration by using a value for x in the given domain
	$\tan^{-1}\left(\frac{1+x}{1-x}\right) = \tan^{-1}x + \frac{\pi}{4}$ hence graph of $y = \tan^{-1}x$, $x < 1$ can be transformed onto the graph of $y = \tan^{-1}\left(\frac{1+x}{1-x}\right)$, $x < 1$ by means of a translation.	**A1**	Correct equation, with terms written in any order, and 'translation'
	[Translation vector =] $\begin{bmatrix} 0 \\ \frac{\pi}{4} \end{bmatrix}$	**B1**	Correct translation vector in exact form
		4	

	Question 9 Total	**7**	

Q	Answer	Marks	Comments
10(a)	$\frac{dy}{dx}=(\sinh 2x)(2\cosh 2x)$	M1	$\frac{dy}{dx}=k(\sinh 2x)(\cosh 2x)$, $k \neq 0$
	$\frac{dy}{dx}=\sinh 4x$	A1	$\frac{dy}{dx}=\sinh 4x$ seen or clearly used
	$1+\left(\frac{dy}{dx}\right)^2=1+\sinh^2 4x=\cosh^2 4x$	A1	Seen or used convincingly
	$S=2\pi\int_0^{0.5}\left(1+0.5\sinh^2 2x\right)\cosh 4x\ dx$	M1	Substitution into correct formula ft their derivative
	$S=2\pi\int_0^{0.5}\left(1+\frac{1}{4}\cosh 4x-\frac{1}{4}\right)\cosh 4x\ dx$	A1	$\sinh^2 2x=\frac{1}{2}(\cosh 4x-1)$ used
	$S=\frac{\pi}{2}\int_0^{0.5}(3+\cosh 4x)\cosh 4x\ dx$	A1	**AG** Be convinced
		6	

Q	Answer	Marks	Comments
10(b)	$S=\frac{\pi}{2}\int_0^{0.5}\left(3\cosh 4x+\frac{1}{2}(\cosh 8x+1)\right)\ dx$	M1	$\cosh^2 4x=\frac{1}{2}(\cosh 8x+1)$ used **PI** by correct integration of $\cosh^2 4x$
	$S=\frac{\pi}{2}\left[\frac{3}{4}\sinh 4x+\frac{1}{2}\left(\frac{1}{8}\sinh 8x+x\right)\right]_0^{0.5}$	A1	Correct integration in hyperbolic form
	$S=\frac{\pi}{2}\left(\frac{3}{4}\sinh 2+\frac{1}{16}\sinh 4+\frac{1}{4}\right)$	A1	**ACF** in terms of hyperbolic functions NMS scores 0/3
		3	

	Question 10 Total	**9**	

Q	Answer	Marks	Comments
11(a)	[Direction vector **v** =] $\begin{bmatrix}3\\-2\\6\end{bmatrix}$	**B1**	Correct direction vector stated or used
	[$\lvert\mathbf{v}\rvert$ =] $\sqrt{3^2+(-2)^2+6^2}$ [= 7]	**M1**	$\sqrt{3^2+(-2)^2+6^2}$ **or** $\sqrt{1^2+0^2+2^2}$ **oe**
	Direction cosines: $\frac{3}{7}$; $-\frac{2}{7}$; $\frac{6}{7}$	**A1**	Correct direction cosines
		3	

Q	Answer	Marks	Comments
11(b)(i)	At point A, $\mathbf{r} = \begin{bmatrix}-2\\2\\-4\end{bmatrix}$		
	$\left(\begin{bmatrix}-2\\2\\-4\end{bmatrix}-\begin{bmatrix}1\\0\\2\end{bmatrix}\right)\times\begin{bmatrix}3\\-2\\6\end{bmatrix}=\begin{bmatrix}-3\\2\\-6\end{bmatrix}\times\begin{bmatrix}3\\-2\\6\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$		
	so A lies on L	**B1**	Correctly verifies that position vector of A satisfies equation of L and states the conclusion
	$\begin{bmatrix}-2\\2\\-4\end{bmatrix}.\begin{bmatrix}1\\2\\-2\end{bmatrix}=-2+4+8=10\neq37$ So A does **not** lie on plane Π	**B1**	Correctly verifies that position vector of A does not satisfy equation of Π and states the conclusion **SC** If verifications both correct but no conclusions then award **SC B1**
		2	

Q	Answer	Marks	Comments
11(b)(ii)	Line through *A* perpendicular to plane Π has equation $\mathbf{r} = \begin{bmatrix}-2\\2\\-4\end{bmatrix} + t\begin{bmatrix}1\\2\\-2\end{bmatrix}$	M1	Finds equation of perpendicular from *A* to the plane; **PI** by general point on the line.
	Meets the plane when $(-2+t)1+(2+2t)2+(-4-2t)(-2)=37$	m1	Solving in order to find a linear equation for the value of *t* at the foot of the perpendicular to Π
	$9t = 27 \Rightarrow t = 3$	A1	
	at *D*, $t = 6$	m1	**Ft** on 2 × c's *t* value at foot of perp
	Posn. vector of $D = \begin{bmatrix}-2\\2\\-4\end{bmatrix} + 6\begin{bmatrix}1\\2\\-2\end{bmatrix} = \begin{bmatrix}4\\14\\-16\end{bmatrix}$		
	Coordinates of *D* (4 , 14, −16)	A1	Correct coordinates for *D*
		5	

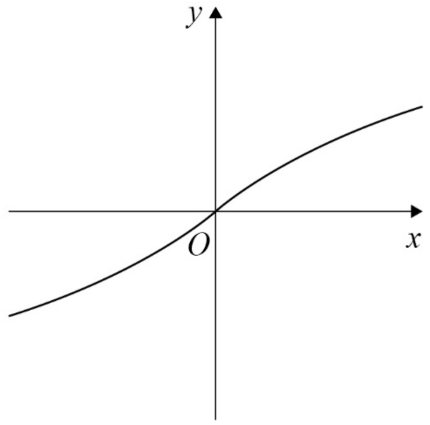
	Question 11 Total	**10**	

Q	Answer	Marks	Comments
12(a)	$\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (3-\tan^2\theta)^2\sec^2\theta \, \mathrm{d}\theta$	M1 B1	$\frac{1}{2}\int r^2 \, [\mathrm{d}\theta]$ **or** $\int_0^{\frac{\pi}{3}} r^2 \, [\mathrm{d}\theta]$ used Correct limits, correct integrand and $\mathrm{d}\theta$ present
	let $u = \tan\theta$, area = $\int_{[0]}^{[\sqrt{3}]} \left(9-6u^2+u^4\right)\mathrm{d}u$	M1	Evidence of valid method to integrate $\tan^n\theta\sec^2\theta$, $n = 2$ **or** 4; could be by inspection. Ignore limits
	area $= \left[9u-2u^3+\frac{1}{5}u^5\right]_{[0]}^{[\sqrt{3}]}$	A1	Integrates $(3-\tan^2\theta)^2\sec^2\theta$ correctly
	$= 9\sqrt{3}-6\sqrt{3}+\frac{9}{5}\sqrt{3}=\frac{24\sqrt{3}}{5}$	A1	**CSO AG**
		5	

Q	Answer	Marks	Comments
12(b)(i)	C_1 : $r\cos\theta = 3 - \tan^2\theta$ $\Rightarrow \quad x = 3 - \frac{y^2}{x^2}$, $y^2 = x^2(3-x)$	M1 A1	Using $r\cos\theta = x$ **or** $\tan\theta = \frac{y}{x}$ **oe** **oe** a correct Cartesian equation for C_1
	at A and B, $x^3 - 4x^2 + 8 = 0$	A1	Obtaining a correct cubic equation when solving C_1 with C_2
	$(x-2)(x^2-2x-4) = 0$, $x = 2$, $x = 1 \pm\sqrt{5}$		
	when $x = 1 + \sqrt{5}$ for C_2 , $y^2 = 2 - 2\sqrt{5} < 0$ **eg** non-real values for y so invalid. and since C_1 has domain $-\frac{\pi}{3} \le \theta \le \frac{\pi}{3}$ **eg** $0 \le x \le 3$, $x = 1 - \sqrt{5}$ is also invalid. When $x = 2$, $y = \pm 2$	E1	Showing that the cubic equation only has one root which gives real values for the coordinates of A and B
	A and B, coordinates (2, 2) and (2, −2)	A1	Previous 4 marks must have been scored
12(b)(i) ALT	$r^2 = 8$ $\sqrt{8}c^3 - 4c^2 + 1 = 0$ where $c = \cos\theta$	M1 A1 A1 E1 A1	Obtaining $r^2 = 8$ as polar eqn of C_2 A correct cubic equation involving θ Further conversion identity to change from polar to Cartesian As in main scheme
		5	

Q	Answer	Marks	Comments
12(b)(ii)	Area of sector *OAB* of circle $C_2 = \frac{1}{2}\left(\sqrt{8}\right)^2 \frac{\pi}{2}$	**B1**	$\frac{1}{2}\left(\sqrt{8}\right)^2 \frac{\pi}{2}$ **oe** exact value
	Area of region bounded by arc *ADB* of C_1 and lines *OA* and *OB*		
	$= \left[9u - 2u^3 + \frac{1}{5}u^5\right]_0^1$ [= 7.2]	**M1**	
	Required area = $7.2 - 2\pi$	**A1**	$7.2 - 2\pi$ **oe** in an exact form
		3	

	Question 12 Total	**13**	

Q	Answer	Marks	Comments
13(a)		B1 B1	Graph only in the 1st and 3rd quadrants, passing through O, and roughly correct shape either in the 1st or 3rd quadrant gradient always positive, increasing in 3rd quadrant but decreasing in the 1st quadrant
		2	

Q	Answer	Marks	Comments
13(b)	$y = \sinh^{-1} x \Rightarrow \sinh y = x$ $\cosh y \dfrac{dy}{dx} = 1$	M1	**oe**
	$\dfrac{dy}{dx} = \dfrac{1}{\pm\sqrt{1+\sinh^2 y}}$	m1	Use of $\cosh^2 y - \sinh^2 y = 1$ Condone missing $\pm$
	Graph of $y = \sinh^{-1} x$ always has positive gradient so $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}}$, $\dfrac{dy}{dx} = \left(1+x^2\right)^{-\frac{1}{2}}$	A1	**AG** Must see $\pm$ and negative sign rejected with a valid reason for doing so otherwise **A0**
13(b) ALT	$y = \ln\left(x+\sqrt{x^2+1}\right) \Rightarrow \dfrac{dy}{dx} = \dfrac{1+\dfrac{0.5\times 2x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}$	M1	
	$\dfrac{dy}{dx} = \dfrac{\sqrt{x^2+1}+x}{\left(\sqrt{x^2+1}\right)\left(x+\sqrt{x^2+1}\right)}$	m1	Multiplying top and bottom by $\sqrt{x^2+1}$ or by $x-\sqrt{x^2+1}$
	$\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2+1}}$, $\dfrac{dy}{dx} = \left(1+x^2\right)^{-\frac{1}{2}}$	A1	**AG**
		3	

Q	Answer	Marks	Comments
13(c)	$\frac{d^2y}{dx^2} = -x(1+x^2)^{-1.5}$	B1	**ACF** a correct expression for $\frac{d^2y}{dx^2}$ **PI**
	$\frac{d^3y}{dx^3} = -(1+x^2)^{-1.5} + 3x^2(1+x^2)^{-2.5}$	M1	Product rule used to find at least one derivative after the 2nd derivative
	when $x = 0$, $\frac{d^3y}{dx^3} = -1 \Rightarrow a = \frac{-1}{3!} = -\frac{1}{6}$ $\left[\frac{d^4y}{dx^4} = \left(9x - 6x^3\right)\left(1+x^2\right)^{-3.5}\right]$	A1	**AG** Must see a correct expression and value at x= 0 for $\frac{d^3y}{dx^3}$ before $a = -\frac{1}{6}$
	$\frac{d^5y}{dx^5} = \left(9 - 72x^2 + 24x^4\right)\left(1+x^2\right)^{-4.5}$ when $x = 0$, $\frac{d^5y}{dx^5} = 9 \Rightarrow b = \frac{9}{120}\left[= \frac{3}{40}\right]$	A1	$b = \frac{9}{120}$ **oe** condone incorrect coefficients of terms in expression for $\frac{d^5y}{dx^5}$ which are 0 when $x = 0$
		4	

Q	Answer	Marks	Comments
13(d)	$\cos 3x = 1-\frac{9}{2}x^2+O(x^4)$	B1	$\cos 3x = 1-\frac{9}{2}x^2+\dots$ seen **or** used
	$\left[\frac{x^2-x\sinh^{-1}x}{(1-\cos 3x)^2}\right]=\frac{x^2-x(x+ax^3+bx^5\dots)}{\left(\frac{9}{2}x^2-O(x^4)\right)^2}$	M1	Substitution of series
	$\lim_{x\to 0}\left[\frac{x^2-x\sinh^{-1}x}{(1-\cos 3x)^2}\right]$ $=\lim_{x\to 0}\left[\frac{-ax^4-bx^6\dots}{\frac{81}{4}x^4-O(x^6)}\right]$ $=\lim_{x\to 0}\left[\frac{-a-bx^2\dots}{\frac{81}{4}-O(x^2)}\right]$ [so the limit exists]	m1	Dividing numerator and denominator by x^4 to get the form $\lim_{x\to 0}\left[\frac{P+O(x^2)}{Q+O(x^2)}\right]$, so the limit exists $=\frac{P}{Q}$ and condone one $O(x^2)$ missing or incorrect power. In place of $O(\ )$ may see equivalent term(s)
	$\left[=\lim_{x\to 0}\left[\frac{\frac{1}{6}-\frac{3}{40}x^2\dots}{\frac{81}{4}-O(x^2)}\right]\right]$ $=\frac{2}{243}$	A1	$\frac{2}{243}$ **A0** if previous 3 marks are not scored
		4	

	Question 13 Total	**13**	