OXFORD
INTERNATIONAL
AQA EXAMINATIONS

INTERNATIONAL A-LEVEL

FURTHER MATHEMATICS

FM03

(9665/FM03) Unit FP2 Pure Mathematics

Mark scheme

January 2020

Version: 1.0 Final

201Xfm03/MS

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Key to mark scheme abbreviations

M	Mark is for method
m	Mark is dependent on one or more M marks and is for method
A	Mark is dependent on M or m marks and is for accuracy
B	Mark is independent of M or m marks and is for method and accuracy
E	Mark is for explanation
√ **or ft**	Follow through from previous incorrect result
CAO	Correct answer only
CSO	Correct solution only
AWFW	Anything which falls within
AWRT	Anything which rounds to
ACF	Any correct form
AG	Answer given
SC	Special case
oe	Or equivalent
A2, 1	2 or 1 (or 0) accuracy marks
–*x* EE	Deduct *x* marks for each error
NMS	No method shown
PI	Possibly implied
SCA	Substantially correct approach
sf	Significant figure(s)
dp	Decimal place(s)

Q	Answer	Marks	Comments
1(a)	Rotation about y-axis through 90°	M1 A1	Rotation identified y-axis and 90° oe (More than one transformation scores 0 marks)
1(b)	$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $\mathbf{A} + \mathbf{B} + \mathbf{B}^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 2 & 0 \end{bmatrix}$	B1 B1ft	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ seen or used for **B** or $\mathbf{B}^{-1}$ If not correct, ft on **A**+2×c's **B**
	Total	**4**	

Q	Answer	Marks	Comments
2	$\int \left(\frac{2x}{x^2+9} - \frac{6}{3x+2}\right)\,dx$ $= \ln(x^2+9) - 2\ln(3x+2)$	B1	Correct integration of $\frac{2x}{x^2+9}$
		B1	Correct integration of $\frac{6}{3x+2}$
	(I=) $\lim_{a\to\infty}\int_0^a \left(\frac{2x}{x^2+9} - \frac{6}{3x+2}\right)\,dx$	M1	∞ replaced by a (oe) and $\lim_{a\to\infty}$ seen or taken at any stage with no remaining lim relating to 0 [Remaining marks are dep on getting only ln terms after integration]
	$= \lim_{a\to\infty}\{\ln(a^2+9) - 2\ln(3a+2)\}$ $-(\ln 9 - 2\ln 2)$		
	$= \lim_{a\to\infty}\left[\ln\left(\frac{a^2+9}{(3a+2)^2}\right)\right] - \ln\left(\frac{9}{4}\right)$	M1	Dealing with the 0 limit correctly and using $\ln P - \ln Q = \ln\left(\frac{P}{Q}\right)$ at least once <u>at any stage</u>
	$= \lim_{a\to\infty}\left[\ln\left(\frac{1+\frac{9}{a^2}}{9+\frac{12}{a}+\frac{4}{a^2}}\right)\right] - \ln\left(\frac{9}{4}\right)$	M1	Writing F(a) **oe** in a suitable form when considering $a \to \infty$
	$\int_0^\infty \left(\frac{2x}{x^2+9} - \frac{6}{3x+2}\right)\,dx$ $= \ln\frac{1}{9} - \ln\frac{9}{4} = \ln\frac{4}{81}$	A1	CSO
	Total	**6**	

Q	Answer	Marks	Comments
3(a)	$(\mathbf{a}\times\mathbf{b})=(-5\mathbf{i}-8\mathbf{j}+\mathbf{k})$	B1	Correct $\mathbf{a}\times\mathbf{b}$ or correct $\mathbf{b}\times\mathbf{a}$
	(Area of triangle=) $=\frac{1}{2}\lvert\mathbf{a}\times\mathbf{b}\rvert=\frac{1}{2}\sqrt{25+64+1}$	M1	Valid method to evaluate $\frac{1}{2}\lvert\mathbf{a}\times\mathbf{b}\rvert$ oe
	$(=\frac{1}{2}\sqrt{90}\ )=\frac{3}{2}\sqrt{10}$	A1	A.G. CSO
3(b)	$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=3(-5)-1(-8)+7(1)$ $=0$	M1	Correct method to evaluate a relevant s.t.p.; ft earlier errors
	$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}=0$ so vectors are coplanar	A1ft	Only ft on wrong sign(s) in c's $\mathbf{a}\times\mathbf{b}$ oe from part (a)
	Total	**5**	

Q	Answer	Marks	Comments
4	When $n=1$, LHS=1, RHS=1 (so formula is true for $n=1$)	B1	Correct values
	Assume formula true for $n=k$ **(*)** integer k, $k \geq 1$ so $\sum_{r=1}^{k+1} r \times 4^{r-1}$	M1	Assumes the result true for $n=k$ and considers $\sum_{r=1}^{k+1} r \times 4^{r-1}$ oe
	$=\frac{1}{9}+\frac{4^k}{9}(3k-1)+(k+1)\times 4^k$		
	$=\frac{1}{9}+\frac{4^k}{9}[3k-1+9(k+1)]$	M1	Grouping the 4^k terms
	$=\frac{1}{9}+\frac{4^k}{9}[12k+8]$	A1	PI by next line
	$=\frac{1}{9}+\frac{4^{k+1}}{9}[3k+2]$ $=\frac{1}{9}+\frac{4^{k+1}}{9}[3(k+1)-1]$	A1	Either
	Hence formula is true for $n=k+1$ **(**)** and since true for $n=1$, formula is true for $n=1,2,3...$ (***) by induction	E1	Must have **(*)** and **(**)** present with 'true for $n=1$' stated at some stage. Previous 5 marks scored and concluding statement **(***)** must clearly indicate that it relates to positive integers eg 'formula true for all $n \geq 1$' is not a precise statement so scores E0
	Total	**6**	

Q	Answer	Marks	Comments
5(a)(i)	Direction vector (**v**=) $\begin{bmatrix} 4 \\ -8 \\ 1 \end{bmatrix}$	B1	Correct direction vector identified
	(| **v**|=) $\sqrt{4^2+(-8)^2+1^2}$ $(=9)$	M1	$\sqrt{4^2+(-8)^2+1^2}$ or $\sqrt{3^2+1^2+2^2}$ oe
	Direction cosines: $\frac{4}{9}$; $-\frac{8}{9}$; $\frac{1}{9}$	A1	Correct direction cosines
5(a)(ii)	$\alpha = \cos^{-1}\left(\frac{4}{9}\right) = 63.6^\circ$	B1ft	Ft on c's $\frac{4}{9}$; ft answer must be correctly rounded
5(b)	$\begin{bmatrix} 3+4t \\ 1-8t \\ 2+t \end{bmatrix}$	B1	A correct position vector of general point on the line seen or used
	12 = $\begin{bmatrix} 3+4t \\ 1-8t \\ 2+t \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ =3+4t+1-8t+2+t	M1	Substitution of c's general point on L into the equation of the plane and scalar product attempted
	$t=-2$	A1	$t=-2$ oe
	(P.V. of pt of intersection=) $\begin{bmatrix} -5 \\ 17 \\ 0 \end{bmatrix}$	A1	$\begin{bmatrix} -5 \\ 17 \\ 0 \end{bmatrix}$ oe
	Total	**8**	

Q	Answer	Marks	Comments
6	$\frac{d^2y}{dx^2}+9y=9x^2+6x+2\cos 3x$		
	Aux. eqn. $m^2+9=0$	M1	PI by correct values of m seen/used
	$(y_{CF}=)A\cos 3x+B\sin 3x$	A1	Correct CF in trig. form
	$(y_{PI}=)\ ax^2+bx+c\quad +dx\sin 3x$	M1 M1	For polynomial form For trig form (If other terms, not in CF or PI, are included in y_{PI}, look to see if their coefficients shown to be 0 later before awarding these **M1** mark(s))
	$(y''_{PI}=)\,2a+6d\cos 3x-9dx\sin 3x$	A1	Correct 2nd derivative
	$9a=9;\ \ 9b=6;\ \ 2a+9c=0;\ 6d=2$	m1	Dep on previous two M marks. Subst. into DE and equating coefficients to form four equations at least two correct. PI by correct values for the coefficients
	$(y_{PI}=)\ x^2+\frac{2}{3}x-\frac{2}{9}\ \ +\frac{1}{3}x\sin 3x$	A1 A1	$x^2+\frac{2}{3}x-\frac{2}{9}$ or correct values for a, b and c; dep on 2nd **M1** mark only $+\frac{1}{3}x\sin 3x$; dep on 3rd **M1** mark only
	$(y_{GS}=)$ $A\cos 3x+B\sin 3x+x^2+\frac{2}{3}x-\frac{2}{9}\ \ +\frac{1}{3}x\sin 3x$	B1ft	c's CF + c's PI but must have exactly two arbitrary constants
	Total	**9**	

Q	Answer	Marks	Comments
7(a)	$x = \tanh y = \dfrac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$ $xe^{y} + xe^{-y} = e^{y} - e^{-y}$ $(x+1)e^{-y} = e^{y}(1-x)$ $\Rightarrow (x+1) = e^{2y}(1-x)$ $e^{2y} = \dfrac{1+x}{1-x} \quad \Rightarrow \quad y = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$ $\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$	 M1 A1 A1	$xe^{y} + xe^{-y} = e^{y} - e^{-y}$ or $xe^{2y} + x = e^{2y} - 1$ A.G. Be convinced. Accept previous line if $y = \tanh^{-1}x$ stated previously **Altn** Reverse order to main scheme: $e^{2y} = \frac{1+x}{1-x}$ **M1**; $x = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}}$ **A1** ; Completion **A1**
7(b)(i)	$\tanh^{-1} x = \dfrac{1}{2}\left[\ln(1+x) - \ln(1-x)\right]$ $= \dfrac{1}{2}\left[x - \dfrac{x^2}{2} + ... - \left(-x - \dfrac{x^2}{2}\right)\right]$ $= \dfrac{1}{2}\left[...(-1)^{r+1}\dfrac{x^r}{r} ... + \dfrac{x^r}{r}\right]$ Coeff. of x^r is $\dfrac{1}{2r}\left[1 + (-1)^{r+1}\right]$	M1 A1	Relevant log law applied and series attempted for both $\ln(1+x)$ and $\ln(1-x)$. PI by correct coefficient of x^r oe Correct coefficient of x^r. Condone if a single x^r is also present with the coefficient.
7(b)(ii)	When $x = 0$, $\dfrac{dy}{dx} = 1$; $\dfrac{1}{3!}\dfrac{d^3y}{dx^3} = \dfrac{1}{3}$; $\dfrac{1}{5!}\dfrac{d^5y}{dx^5} = \dfrac{1}{5}$; $\dfrac{1}{7!}\dfrac{d^7y}{dx^7} = \dfrac{1}{7}$ When $x = 0$, $\left(\dfrac{dy}{dx} + \dfrac{d^3y}{dx^3} + \dfrac{d^5y}{dx^5} + \dfrac{d^7y}{dx^7}\right)$ $= 1+2+24+720 = 747$	M1 A1	Comparing coefficients of x, x^3, x^5 and x^7 from (b)(i) with the general Maclaurin's series oe by direct differentiations 747

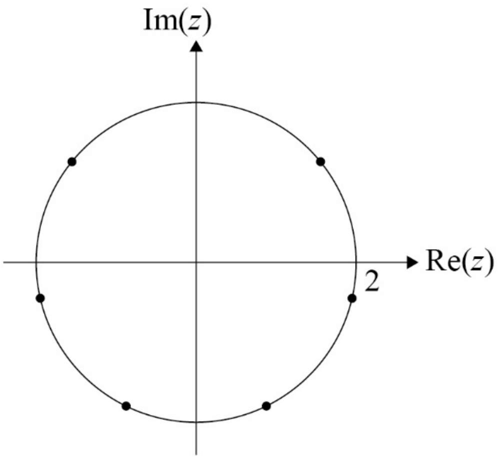
	Total	**7**	

Q	Answer	Marks	Comments
8(a)	$\det \mathbf{A} = 1\left(k^2 - 12\right) - 2(k-8) - 1(3-2k)$	**M1**	Correct method to expand det $\mathbf{A}$ by row or column
	$\det \mathbf{A} = k^2 + 1$	**A1**	
	Since k is real, $k^2 \geq 0$ so $(\det \mathbf{A}) \neq 0$ so $\mathbf{A}$ is non-singular	**E1**	Ft only on det $\mathbf{A} = k^2 + c$, where c is a positive integer. 'det **A** > 0 so **A** is non-singular' is **E0**; we must see reference to non-zero with justification
8(b)	Cofactor matrix $\begin{bmatrix} k^2-12 & -k+8 & -2k+3 \\ -2k-3 & k+2 & 1 \\ k+8 & -5 & k-2 \end{bmatrix}$	**M1**	One complete row or column correct
		A2,1,0	**A2** all 9 correct; else **A1** at least 6 correct
	Inverse matrix $\mathbf{A}^{-1} =$	**M1**	Transpose of their cofactors with no more than one further error <u>**and**</u> division by their det $\mathbf{A} \neq 0$
	$\dfrac{1}{k^2+1}\begin{bmatrix} k^2-12 & -2k-3 & k+8 \\ -k+8 & k+2 & -5 \\ -2k+3 & 1 & k-2 \end{bmatrix}$	**A1ft**	<u>Only</u> ft on their det $\mathbf{A}$ from part (a) provided their det $\mathbf{A}$ is non-zero for all real values of k
8(c)	$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1}\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$		
	$= \dfrac{1}{k^2+1}\begin{bmatrix} k^2-12-6k-9+6k+48 \\ -k+8+3k+6-30 \\ -2k+3+3+6k-12 \end{bmatrix}$	**M1**	$\mathbf{A}^{-1}\mathbf{v}$ for c's $\mathbf{A}^{-1}$ with at least one ft component correct
		A1ft	At least two ft components correct
	$x = \dfrac{k^2+27}{k^2+1}$ $y = \dfrac{2k-16}{k^2+1}$ $z = \dfrac{4k-6}{k^2+1}$	**A1**	All correct
			NB 0/3 scored if $\mathbf{A}^{-1}$ not used.
	Total	**11**	

Q	Answer	Marks	Comments
9(a)(i)	Given $\alpha+\beta=0$ $\alpha+\beta+\gamma+\delta=-\frac{1}{m} \quad \Rightarrow \gamma+\delta=-\frac{1}{m}$ (*)	E1	
9(a)(ii)	$\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=\frac{m+n}{m}$ (**)	M1	or $\sum\alpha\beta=\frac{m+n}{m}$
	From (**), $(\alpha+\beta)(\gamma+\delta)+\alpha\beta+\gamma\delta=\frac{m+n}{m}$ so $\alpha\beta+\gamma\delta=\frac{m+n}{m}$	A1	
	$\alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\alpha\gamma\delta=\frac{1}{m}$ (#); $\alpha\beta\gamma\delta=\frac{n}{m}$ (# #)	M1	Either (#) or (# #) or both $\sum\alpha\beta\gamma=\frac{1}{m}$ and $\sum\alpha\beta\gamma\delta=\frac{n}{m}$
	From (#) $\alpha\beta(\gamma+\delta)=\frac{1}{m}$ so $\alpha\beta(-\frac{1}{m})=\frac{1}{m} \Rightarrow \alpha\beta=-1$	A1	
	Sub into (# #) gives $\gamma\delta=-\frac{n}{m}$	A1	
	$\alpha\beta+\gamma\delta=\frac{m+n}{m}=1+\frac{n}{m}$ so $-1-\frac{n}{m}=1+\frac{n}{m}, \quad -2=\frac{2n}{m} \quad \Rightarrow n=-m$	A1	AG be convinced Condone if left as $m=-n$

9(b)	$\alpha+\beta=0$, and $\alpha\beta=-1$ so a quadratic factor is x^2-1 $mx^4+x^3-x-m=0$ $(x^2-1)(mx^2+x+m)=0$ Roots are 1, -1, $\dfrac{-1\pm\sqrt{1-4m^2}}{2m}$ 4 distinct real roots $\Rightarrow 4m^2<1$, $m\neq 0$ ie $-\dfrac{1}{2}<m<0$, $0<m<\dfrac{1}{2}$	**M1** **M1** **A1** **A1**	Finding a quadratic factor **PI** Finding other quadratic factor by division or by sum and product of roots method. Correct four roots or $1-4m^2>0$ oe
	Total	**11**	

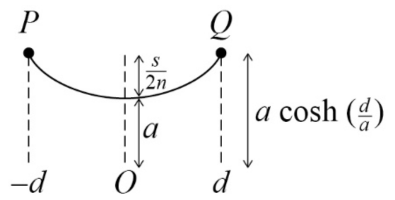
Q	Answer	Marks	Comments
10(a)	$\frac{dy}{dx}+\frac{2}{x}y=\frac{\cos x}{x}$		
	I.F. is $\exp\left(\int \frac{2}{x}(dx)\right) = e^{2\ln x}$	M1	Identified and integration attempted **PI**
	(I.F.) $= x^2$	A1	Seen or used
	$\frac{d}{dx}\left[x^2 y\right]= x\cos x$; $x^2y=\int x\cos x\,(dx)$	M1	Either
	$x^2y = x\sin x - \int \sin x\,(dx)$	A1	PI by next line
	$x^2y = x\sin x+\cos x+p$; $y = x^{-2}(x\sin x+\cos x+p)$	A1	Either
10(b)(i)	As $x\to 0$, $y\to\frac{x\left(x-O(x^3)\right)+1-0.5x^2+...+p}{x^2}$	M1	$\sin x = x$ (…) or $\cos x = 1-0.5x^2$ (…) substituted in c's GS
		B1	Both $\sin x = x$ (…) and $\cos x = 1-0.5x^2$ (…) substituted in c's GS
	$y\to\frac{1+p}{x^2}+0.5+O(x^2) \Rightarrow p=-1$	A1ft	Ft on numerical and sign errors in c's GS
	$\Rightarrow y\to 0.5$ as $x\to 0$ $\Rightarrow k=0.5$	A1	Correct value for k dep. on p found so that no term $\to\infty$ as $x\to 0$
10(b)(ii)	At st. pts. $\frac{dy}{dx}=0$ subst into given DE $\Rightarrow y=0.5\cos x$	M1	No more than one numerical/sign error in finding y as a multiple of $\cos x$ when $\frac{dy}{dx}=0$
	Since $k=0.5$, all stationary points of curve *C* lie on the curve $y=k\cos x$ so the student is correct	A1ft	Ft c's value for k but conclusion must be related to comparison of c's k with 0.5
	Total	**11**	

Q	Answer	Marks	Comments
11(a)	$128\,\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{2}\right)}$	B1; B1	$r = 128$; $\theta = -\frac{\pi}{2}$
11(b)	$r = \sqrt[7]{128} = 2$ Use of de Moivre: $\text{c's}\left(-\frac{\pi}{2}\right) \div 7$ $\theta = -\frac{\pi}{14} + \frac{2k\pi}{7}$, k=0,±1,±2,±3 (7 roots of z^7+128i = 0 are) $2\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{14}\right)}$; $2\mathrm{e}^{\mathrm{i}\left(\frac{3\pi}{14}\right)}$; $2\mathrm{e}^{\mathrm{i}\left(\frac{\pi}{2}\right)}(=2\mathrm{i})$; $2\mathrm{e}^{\mathrm{i}\left(\frac{11\pi}{14}\right)}$ $2\mathrm{e}^{\mathrm{i}\left(-\frac{5\pi}{14}\right)}$; $2\mathrm{e}^{\mathrm{i}\left(-\frac{9\pi}{14}\right)}$; $2\mathrm{e}^{\mathrm{i}\left(-\frac{13\pi}{14}\right)}$	B1 M1 A1 A1	$r = 2$ If incorrect, ft on c's $-\frac{\pi}{2}$ in part **(a)** 7 correct values for θ; mod 2π CAO
11(c)(i)		B1ft B1 B1	Clear indication that the six roots lie on a circle of radius 2; ft c's r value in part **(b)** Points shown on Argand diagram: Six points in the correct quadrants. Pairs of points having sym about the Im axis with no pair of points having sym about the Re axis.

11(c)(ii)	$2e^{i\left(-\frac{\pi}{14}\right)}, 2e^{i\left(-\frac{13\pi}{14}\right)}$ and $2e^{i\left(\frac{3\pi}{14}\right)}, 2e^{i\left(\frac{11\pi}{14}\right)}$ and $2e^{i\left(-\frac{9\pi}{14}\right)}, 2e^{i\left(-\frac{5\pi}{14}\right)}$	**M1**	Choosing three pairs of c's roots whose products are real;
	Factors: $[z^2 - 2(e^{i\left(-\frac{\pi}{14}\right)} + e^{i\left(-\frac{13\pi}{14}\right)})z - 4]$; $[z^2 - 2(e^{i\left(\frac{3\pi}{14}\right)} + e^{i\left(\frac{11\pi}{14}\right)})z - 4]$; $[z^2 - 2(e^{i\left(-\frac{9\pi}{14}\right)} + e^{i\left(-\frac{5\pi}{14}\right)})z - 4]$	**A1ft**	Two correct ft on c's r value in **(b)** in form shown or better eg $[z^2 + 2(e^{i\left(\frac{\pi}{14}\right)} - e^{-i\left(\frac{\pi}{14}\right)})z - 4]$; $[z^2 - 2(e^{i\left(\frac{3\pi}{14}\right)} - e^{-i\left(\frac{3\pi}{14}\right)})z - 4]$; $[z^2 + 2(e^{i\left(\frac{5\pi}{14}\right)} - e^{-i\left(\frac{5\pi}{14}\right)})z - 4]$
		M1	Correct attempt to find two correct values for q in factors $z^2 + i(p\sin(q\pi))z + t$ where $\lvert q\rvert < \frac{1}{2}$
	$Q(z) = \left[z^2 + i(4\sin\frac{\pi}{14})z - 4\right]\left[z^2 - i(4\sin\frac{3\pi}{14})z - 4\right]\left[z^2 + i(4\sin\frac{5\pi}{14})z - 4\right]$	**A1**	A correct product of three quadratic factors in the required form.
	Total	**13**	

Q	Answer	Marks	Comments
12(a)(i)	When $\theta = \frac{7\pi}{6}$, $r = \sin(\pi) = 0$ $\Rightarrow$ circle passes through the pole O	**B1**	Use of either $\theta = \frac{7\pi}{6}$ or $\theta = \frac{\pi}{6}$ to give $r = 0$
12(a)(ii)	(Area of C_2) $= \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{7\pi}{6}} \sin^2\left(\theta - \frac{\pi}{6}\right)(d\theta)$	**M1**	A correct definite integral for the area of C_2. **PI** if limits missing but seen later
	$= \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{7\pi}{6}} [1 - \cos 2\left(\theta - \frac{\pi}{6}\right)](d\theta)$	**M1**	Expressing the integrand in terms of $\cos 2\left(\theta - \frac{\pi}{6}\right)$ **oe**
	$= \frac{1}{4}\left[\theta - \frac{1}{2}\sin\left(2\theta - \frac{\pi}{3}\right)\right]_{\frac{\pi}{6}}^{\frac{7\pi}{6}} = \frac{6\pi}{24} = \frac{\pi}{4}$	**A1**	CSO

12(b)(i)	L: $\sqrt{3}\,y = 1 - x$ (Polar eqn of L) $\sqrt{3}\,r\sin\theta = 1 - r\cos\theta$ $\dfrac{1}{\sqrt{3}\sin\theta + \cos\theta} = \dfrac{2}{3 + 2\cos\theta}$ $\Rightarrow \sin\theta = \dfrac{\sqrt{3}}{2}$ $\Rightarrow \theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}$ When $\theta = \frac{\pi}{3}$, $r = \frac{1}{2}$; When $\theta = \frac{2\pi}{3}$, $r=1$ $\sin\left(\frac{2\pi}{3} - \frac{\pi}{6}\right) = \sin\frac{\pi}{2} = 1$; $\left(1, \frac{2\pi}{3}\right)$ on C_2 $\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$; $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ on C_2 Points $\left(1, \frac{2\pi}{3}\right)$ and $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ satisfy polar equations of L, C_1 and C_2, and since $OP>OQ$, $P\left(1, \dfrac{2\pi}{3}\right)$, $Q\left(\dfrac{1}{2}, \dfrac{\pi}{3}\right)$ are the required points of intersection.	M1 M1 A1 A1 A1	Use of either $y = r\sin\theta$ or $x = r\cos\theta$ Equating r s for L and C_1 and attempt to find a value for a single trig term **oe** Forming a correct relevant quadratic equation and solving to find Cartesian coordinates At least 3 of the 4 polar values Verifying that both $\left(1, \frac{2\pi}{3}\right)$ and $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ satisfy eqn of C_2 Identifying correct P and Q plus a relevant concluding statement
12(b)(ii)	From **(a)(ii)**, radius of circle C_2 = 0.5 From **(a)(i)** and **(b)(i)** O and P are points on C_2 and length of OP=1=2×radius so OP is a diameter	E1 E1	Accept any valid explanations but must include O and P being points on C_2 when referring to the length of OP
12(c)	$\tan[\pi/2-(2\pi/3-\pi/3)] = \tan(\pi/6)$ $y = x\tan\left(\dfrac{\pi}{6}\right) + c$ $P\left(1\cos\frac{2\pi}{3}, 1\sin\frac{2\pi}{3}\right)$ $y = \dfrac{x+2}{\sqrt{3}}$	M1 A1 B1ft A1	Using relevant detail(s) from part(s) **(b)** in attempt to find the gradient of the tangent at O or P Equation of tangent at P with a correct gradient c's Polar coordinates of P correctly converted to Cartesian form **oe** A correct Cartesian equation of tangent at P with all trig terms evaluated
	Total	**15**	

Q	Answer	Marks	Comments
13(a)	$\frac{dy}{dx} = \sinh\left(\frac{x}{a}\right)$	B1	Correct differentiation
	(s=) $\int_{-d}^{d} \sqrt{1+\sinh^2\left(\frac{x}{a}\right)}\ dx$	M1	Correct ft integral
	(s=) $\int_{(-d)}^{(d)} \cosh\left(\frac{x}{a}\right)\ dx$	A1	
	$= \left[a\sinh\left(\frac{x}{a}\right)\right]_{-d}^{d}$ $= a\sinh\left(\frac{d}{a}\right) - a\sinh\left(-\frac{d}{a}\right) = 2a\sinh\left(\frac{d}{a}\right)$	A1	A.G. be convinced
13(b)(i)		E1	Sketch of the chain as a cosh curve with sufficient detail eg lowest pt (0, a) of cosh curve being a distance $\frac{s}{2n}$ below PQ and height of PQ above x-axis **oe** being $a\cosh\left(\frac{d}{a}\right)$, used to justify $a + \frac{s}{2n} = a\cosh\left(\frac{d}{a}\right)$
13(b)(ii)	$a + \frac{s}{2n} = a\cosh\left(\frac{d}{a}\right) = a\sqrt{1+\sinh^2\left(\frac{d}{a}\right)}$	M1	$\cosh\left(\frac{d}{a}\right) = \sqrt{1+\sinh^2\left(\frac{d}{a}\right)}$ used
	$= a\sqrt{1+\left(\frac{s}{2a}\right)^2} = \sqrt{a^2+\frac{s^2}{4}}$	A1	A.G. be convinced

13(b)(iii)	$a^2 + \frac{as}{n} + \frac{s^2}{4n^2} = a^2 + \frac{s^2}{4}$	M1	Squaring both sides
	$\frac{as}{n} = \frac{s^2}{4n^2}(n^2 - 1) \Rightarrow a = \frac{s}{4n}(n^2 - 1)$	A1	$a = \frac{s}{4n}(n^2 - 1)$
	$\frac{2a\sinh\left(\frac{d}{a}\right)}{a\cosh\left(\frac{d}{a}\right)} = \frac{s}{a + \frac{s}{2n}} \Rightarrow 2\tanh\left(\frac{d}{a}\right) = \frac{s}{a + \frac{s}{2n}}$	M1	Identity $\frac{\sinh x}{\cosh x} = \tanh x$ used
	$\tanh\left(\frac{d}{a}\right) = \frac{2n}{n^2 + 1}$	A1	
	$\frac{d}{a} = \tanh^{-1}\left(\frac{2n}{n^2+1}\right) = \frac{1}{2}\ln\left[\frac{1 + \frac{2n}{n^2+1}}{1 - \frac{2n}{n^2+1}}\right]$	M1	$\tanh^{-1}\left(\frac{2n}{n^2+1}\right) = \frac{1}{2}\ln\left[\frac{1+\mathrm{f}(n)}{1-\mathrm{f}(n)}\right]$
	$\frac{d}{a} = \frac{1}{2}\ln\left[\frac{(n+1)^2}{(n-1)^2}\right] = \ln\left(\frac{n+1}{n-1}\right)$	A1	
	$PQ = \frac{s}{2n}(n^2 - 1)\ln\left(\frac{n+1}{n-1}\right)$	A1	A.G. Be convinced
	Total	**14**	

13(b)(iii) ALT	$a^2+\frac{as}{n}+\frac{s^2}{4n^2}=a^2+\frac{s^2}{4}$	**M1**	Squaring both sides
	$\frac{as}{n}=\frac{s^2}{4n^2}(n^2-1)\Rightarrow a=\frac{s}{4n}(n^2-1)$	**A1**	$a=\frac{s}{4n}(n^2-1)$
	$PQ=2d=2a\sinh^{-1}\left(\frac{s}{2a}\right)$	**(M1)**	$PQ=2a\sinh^{-1}\left(\frac{s}{2a}\right)$ oe or $PQ=2a\cosh^{-1}\left(1+\frac{s}{2na}\right)$ oe
	$=\frac{s}{2n}(n^2-1)\sinh^{-1}\left(\frac{2n}{n^2-1}\right)$	**(A1)**	oe
	$=\frac{s}{2n}(n^2-1)\ln\left[\frac{2n}{n^2-1}+\sqrt{1+\frac{4n^2}{(n^2-1)^2}}\right]$	**(M1)**	$\sinh^{-1}[\mathrm{f}(n)]=\ln\left[\mathrm{f}(n)+\sqrt{1+\{\mathrm{f}(n)\}^2}\right]$ or $\cosh^{-1}[\mathrm{f}(n)]=\ln\left[\mathrm{f}(n)+\sqrt{\{\mathrm{f}(n)\}^2-1}\right]$
	$=\frac{s}{2n}(n^2-1)\ln\left[\frac{2n+\sqrt{(n^2+1)^2}}{n^2-1}\right]$ $=\frac{s}{2n}(n^2-1)\ln\left[\frac{(n+1)^2}{(n+1)(n-1)}\right]$	**(A1)**	oe
	$PQ=\frac{s}{2n}(n^2-1)\ln\left(\frac{n+1}{n-1}\right)$	**(A1)**	A.G. Be convinced
	Total	**14**	